B.Sc. II nd Semester (CBCS)

Maxwell's Equations and Electromagnetic wave propagation

Equation of continuity of current

Law of conservation of charge holds good for any situation such as motion of charge interconversion of particles and can be stated as follows.

In all processes associated with the motion of charge carriers the charge is always conserved

Integral form of charge conservation law

Assuming charge conservation as an experimental fact, we can express it as the statement that the charge in a certain volume V can change only if charge flows in to or out of it. i.e for closed conducter S bounding the volume

$$\frac{\partial}{\partial t}\int_V \rho dv = -\oint j ds . \qquad \text{—— (1)}$$

where ρ is the charge density ($dq = \rho dv$) and j is current density per unit area of cross section

The left hand side of equation (1) defines the rate of variation of charge in the volume, and the right side is the strength of current flowing through the surface bounding inside the volume decreases, the current density is directed outwards from the volume, It should be under closed surface consequently, vector S in above relation is directed along the outward normal to the surface

Differential form of charge conservation law

Volume V and surface area S in Eqn (1) do not change with time consequently the time derivative on the left side of Eqn (1) can be included in the integral, on the other hand the right hand side of this ~~equation~~ equality can be transfered in to a volume integral in accordance with Gauss theorem

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$$\therefore \frac{\partial}{\partial t}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t} dV$$

and $\oint j\, ds = \int_V \text{div}\, j\, dV$

Transposing all terms in eqn ① we get

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \text{div}\, j\, dV.$$

or $$\int_V \left[\frac{\partial \rho}{\partial t} + \text{div}\, j\right] dV = 0 \qquad \text{——} ②$$

This equality is valid for any volume, The proof that the integral is zero can be obtained by contradiction. If the integral is not equal to zero at some point, we can take for V a small volume around this point in which sign of the integrand remains unchanged, This however is in contradiction to eqn ②. consequently the integrand is equal to zero at all points, In this case

$$\frac{\partial \rho}{\partial t} + \text{div}\, j = 0.$$

or $$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0. \qquad \text{——} ③$$

This is the differential form of the law of charge conservation It is also called the continuity equation

Displacement current

There is no direct current in a circuit containing a capacitor, while an alternating current can flow in it. The magnitude of a quality / stationary current is same for all series-connected elements of the circuit, The conduction current due to motion of electrons can not pass through a capacitor as its plates are separated by a dielectric. As the current does pass through the capacitor, so we have to conclude that in a capacitor a certain process closes the conduction current i.e it enables in some way the charge exchange between the capacitor plates without actually transporting a charge between the plates The current associated with this process is called displacement current

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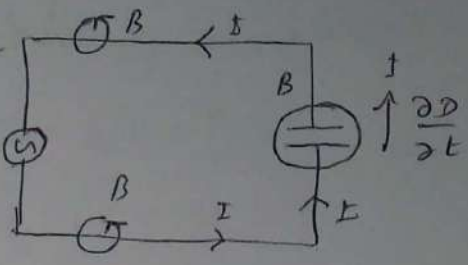
Let us consider an a.c. circuit having a parallel plate capacitor as shown in fig. between the capacitor plate an electric field strength



$E = \frac{\sigma}{\varepsilon}$ exists, where σ is the charge density on the plates and ε is the permitivity of the medium between the plates, the electric displacement vector between two capacitor plates is given by

$$D = \sigma = \frac{Q}{S}$$

where Q is the charge on each plates and S is the surface area of the plate, The current in the circuit is

$$I = \frac{\partial Q}{\partial t}$$ Hence it follows that

$$Id = S \frac{\partial D}{\partial t} \quad \text{——— (1)}$$

$$jd = \frac{\partial D}{\partial t} \quad \text{——— (2)}$$

From the above we observe that a change in the electric displacement or electric field ($D = \varepsilon E$) between two capacitor plates is the process that closes the conduction current in the circuit ~~it~~ in Eqn (1) the quantity I has a subscript 'd' (displacement) in order to emphasize that this quantity is not the conduction current between the plates

The concept of displacement current was a great theoritical discovery as it rendered the relation $\nabla \times H = J$ between the current and magnetic field consistent with the continuity equation, This concept also ~~leads~~ lead to the new concept of induction effect, according to which a time varying electric field should give rise to magnetic effect as a time-varying magnetic field gives rise to electric field, further it helped to retain the notation that the flow of current in a circuit is continuous

Magnitude of Displacement current

If Q is the charge given to one of the plate of the capacitor at any instant and S is the area of plates

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the electric field between the plates of the capacitor is given by

$$E = \frac{Q}{\varepsilon S} \quad \text{and} \quad D = \varepsilon_0 E = \frac{Q}{S}.$$

Differentiating above relation w.r.t to t.

$$\frac{\partial D}{\partial t} = \frac{1}{S}\frac{dQ}{dt} = \frac{i}{S}.$$

the rate of transfer of charge $\frac{dQ}{dt} = i$, the current flowing in the circuit, Hence

$$\frac{\partial D}{\partial t} = \frac{i}{S} \int A_s j_d = \frac{dD}{dt}, \quad j_d = \frac{i}{S}.$$

$$j_d s = i \quad \text{or} \quad i_d = i$$

Hence the displacement current in the gap between the capacitor plates is equal to the conduction current flowing in the circuit, It may be noted that the displacement current in vacuum is not a current in the sense that it describes the motion of some charged particles, But it is a current- in the sense it has magnetic field associated with it which is an essential properties of electric current.

Maxwell's Equations

The Ampere's law given by the equation $\nabla \times B = \mu_0 j$ holds good only for time independent fields and does not holds good for time varying fields and this rendered the above equations inconsistant which makes the set of four equations inconsistent, To overcome this difficulty maxwell modified the above equations by introducing the concept of displacement current, This concept is as follows

The differential form of Ampere's law is

$$\nabla \times B = \mu_0 j$$

using the relation between magnetic induction (B) and magnetic field intensity (H) i.e.

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$B = \mu_0 H$ we have

$$\frac{1}{\mu_0}(\nabla \times B) = j$$

$$\nabla \times H = j$$

Taking divergence on both side we get

$$\nabla \cdot \nabla \times H = \nabla \cdot j$$

As the divergence of curl of a vector is always zero. we have

$$\nabla \cdot j = 0.$$

The above relation shows that the net flux of current out of any closed surface is zero which is the equation of continuity for steady fields and does not apply to time varying fields. The above equation can be modified to a form, whose divergence is zero, irrespective of the nature of the field. from Gauss's law we have

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}.$$

Differentiating above eqn w.r.to t.

$$\nabla \cdot \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial t}$$

$$\varepsilon_0 \nabla \cdot \frac{\partial E}{\partial t} = \frac{\partial \rho}{\partial t}$$

Adding $\nabla \cdot j$ on both sides we get

$$\nabla \cdot j + \varepsilon_0 \nabla \cdot \frac{\partial E}{\partial t} = \nabla \cdot j + \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot j + \varepsilon_0 \nabla \cdot \frac{\partial E}{\partial t} = 0$$

As $\nabla \cdot j + \frac{\partial \rho}{\partial t} = 0$, according to general equation of continuity

$$\nabla \cdot \left[j + \varepsilon_0 \frac{\partial E}{\partial t} \right] = 0.$$

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Substituting $D = \varepsilon_0 E$ we get

$$\nabla \cdot \left[j + \frac{\partial D}{\partial t} \right] = 0.$$

Replacing j by $j + \frac{\partial D}{\partial t}$ in Ampere's law [Eqn ①] we get

$$\nabla \times H = j + \frac{\partial D}{\partial t}$$

The term added to j i.e. $\frac{\partial D}{\partial t}$ is called displacement current and has the same effect as the true current but it comes in to play or is effective only when E is a time varying field

The following Four equations

$$\nabla \cdot E = \rho/\varepsilon_0 \quad \text{——} ①$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{——} ②$$

$$\nabla \cdot B = 0 \quad \text{——} ③$$

$$\nabla \times H = \left[j + \frac{\partial D}{\partial t} \right] \quad \text{——} ④$$

The equations ① ②, ③ and ④ are called maxwell's equations

Poynting Vector: Energy Density in Electromagnetic field:

Consider an electromagnetic wave travelling along x-axis with the magnetic and electric fields H and E confined in the transmission planes along z and y-axis respectively. This wave thus propagates along the direction of propagation vector $(E \times H)$

Taking divergence of poynting vector in free space

$$\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E (\nabla \times H)$$

$$= -H \frac{\partial B}{\partial t} - E \frac{\partial D}{\partial t}$$

by using Maxwell ④ equation

$$= -\left[E \frac{\partial D}{\partial t} + H \frac{\partial B}{\partial t} \right]$$

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$$= -\left[\epsilon_0 E \frac{\partial E}{\partial t} + \mu_0 H \frac{\partial H}{\partial t}\right]$$

by using relations $D = \epsilon_0 E$ and $B = \mu_0 H$

$$= -\left[\frac{1}{2}\epsilon_0 2E \frac{\partial E}{\partial t} + \frac{1}{2}\mu_0 2H \frac{\partial H}{\partial t}\right]$$

$$= -\left[\frac{1}{2}\epsilon_0 \frac{\partial (E)^2}{\partial t} + \frac{1}{2}\mu_0 \frac{\partial (H)^2}{\partial t}\right]$$

$$= -\frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right]$$

Considering that Surface S' bounds volume V and integrating the above equation over the volume V, we get

$$\int_V \nabla \cdot (E \times H) dV = -\frac{\partial}{\partial t}\int_V \left[\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right] dV.$$

On applying divergence theorem to the left hand side term of above equation we get

$$\int_S (E \times H) \cdot ds = -\frac{\partial}{\partial t}\int_V \left[\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right] dV \quad \text{—①}$$

The terms of the right hand side of within the integral of eqn ① represents the sum of energies of electric and magnetic fields. Hence right hand side of the eqn ① represents the amount of energy transferred over the volume in one second. i.e it represents the rate of flow of energy over the volume V.

Energy associated with electric field $U_s = \frac{\epsilon_0 E^2}{2}$ and that of magnetic field

$$U_m = \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\frac{B^2}{\mu_0}$$

As $U_m = \frac{B^2}{2\mu_0} = \frac{\left(\frac{E}{c}\right)^2}{2\mu_0} = \frac{1}{2}\epsilon_0 E^2 = U_E$ (as $E/B = c$

and $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

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which shows that instanteneous energy density associated with the magnetic field is equal to the instanteneous energy density associated with electric field i.e. energy is equally shared by the two fields.

The vector $P(=E\times H)$ in Eqn ① is interpreted as representing the amount of field energy passing through unit area of surface in unit time normally to the direction of flow of energy, This statement is termed as poynting's theorem and the vector 'P' is called poynting vector

Hence the poynting vector integrated over a closed surface i.e. the term $\oint_S (E\times H)\,ds$ represents the rate of flow of energy over the surface S' enclosing a volume V. From above we can say poynting vector represents the flow of energy over a given volume, vector 'P' has the dimensions of $watt/m^2$ and direction of flow is $\perp^r$ to E and H. i.e. in the direction of the vector $E\times H$.

Electromagnetic waves in free space

Let us consider a region where charge density ρ. current density j are both zero. maxwell's Eqn reduced to

$$\left.\begin{aligned} \nabla\cdot E &= 0 \quad \text{——— (1)i} \\ \nabla\times E + \frac{\partial B}{\partial t} &= 0 \quad \text{——— (1)ii} \\ \nabla\cdot B &= 0 \quad \text{——— (3)(iii)} \\ \nabla\times H &= \frac{\partial D}{\partial t} \quad \text{——— (4)(iv)} \end{aligned}\right\} \quad \text{——— (1)}$$

D.ff. Eqn ② we get

$$\nabla\times\frac{\partial E}{\partial t} + \frac{\partial^2 B}{\partial t^2} = 0 \quad \text{——— (2)}$$

From eqn (iv) of Eqn. ① we get on taking curl.

$$\nabla\times\nabla\times H = \nabla\times\frac{\partial D}{\partial t} \quad (\text{as } D = \epsilon_0 E)$$

$$\nabla\times\nabla\times H = \epsilon_0\nabla\times\frac{\partial E}{\partial t} \quad \text{——— (3)}$$

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Substituting from eqn (3) for $\nabla \times \frac{\partial E}{\partial t}$ we get-

$$\nabla \times \nabla \times H = -\epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\nabla(\nabla \cdot H) - \nabla^2 H = -\mu_0 \epsilon_0 \frac{\partial^2 H}{\partial t^2} \quad (\text{As } B = \mu_0 H)$$

Since $\nabla \cdot B = 0$, so, $\nabla \cdot H = 0$, from relation (iii) of eqn (1) hence we have

$$\nabla^2 H = \mu_0 \epsilon_0 \frac{\partial^2 H}{\partial t^2} \qquad \text{——— (4)}$$

Similarly on diff. both sides of eqn (iv) of eqn (1) w.r.to t. and proceeding in the same way. we get-

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad \text{——— (5)}$$

Eqn (4) & eqn (5) shows that each component H & E gives the general wave equation

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0.$$

where $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is constant, $\epsilon_0 = 8.854 \times 10^{-12}$ F/m.

$\mu_0 = 4\pi \times 10^{-7}$ H/m.

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}}$$

$$= 2.998 \times 10^8 \text{ m/s}$$

This is the speed of light in vacuum or free space From the above one can conclude that the field vectors E and H can be propagated as waves in free space and they travel with the speed of light-

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Transverse Nature of EM Waves

Let us assume that the wave is travelling along x-axis i.e. it is a function of only x and t i.e. E and H depend on x and t only and are independent of y and z i.e.

$$E = E(x, t)$$

$$\frac{\partial E_y}{\partial y} = 0, \text{ and } \frac{\partial E_z}{\partial z} = 0. \quad \text{——— (8)}$$

Now using Maxwell's eqn (i) of eqn (1) i.e. $\nabla \cdot E = 0$.

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0.$$

adding eqn (8) in above we get

$$\frac{\partial E_x}{\partial x} = 0.$$

From above eqn it is clear that E_x is not a function of x i.e. E_x does not depend on the direction of propagation of EM waves, then $E_x = 0$. ——— (9)

Thus the electric field vector E of the wave does not have any component along x-axis but can have components only along y-axis and z-axis

This implies that E can not have any Longitudinal component but can have only transverse components. Let E be taken parallel to y-axis of the co-ordinates system

The electric field E then will have only y-component and no-z-component i.e.

or $E = jE_y(x, t)$ ——— (10)

From maxwell's eqn (2)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$= i\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + j\left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right] + k\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]$$

$$= -\left[i\frac{\partial B_x}{\partial t} + j\frac{\partial B_y}{\partial t} + k\frac{\partial B_z}{\partial t}\right]$$

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Equating the components along -x, -y, and -z axis

$$\left.\begin{aligned}\left[\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right] &= -\frac{\partial B_x}{\partial t}\\ \left[\frac{\partial E_x}{\partial z}-\frac{\partial E_x}{\partial z}\right] &= -\frac{\partial B_y}{\partial t}\\ \left[\frac{\partial E_y}{\partial z}-\frac{\partial E_x}{\partial y}\right] &= -\frac{\partial B_z}{\partial t}\end{aligned}\right\} \quad \text{——— (11)}$$

By assuming.

$E_x = 0$, $E_z = 0$, $\frac{\partial E_y}{\partial y} = 0$ and $\frac{\partial E_z}{\partial z} = 0$

put these in eqn (11) we get-

$\frac{\partial B_x}{\partial t} = 0$, $\frac{\partial B_y}{\partial t} = 0$, and $\frac{\partial E_y}{\partial t} = -\frac{\partial B_z}{\partial t}$

which shows B_x and B_y are constant w.r to time As we are interested only in varying fields, so. may can be taken as zero

or $B_x = 0$, $B_y = 0$. ——— (12)

$H_x = 0$, $H_y = 0$.

and $\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x}$ ——— (13)

From eqn (10) and (12) we observe that a plane electromagnetic wave travelling in air is transverse in nature

Vector E and Vector H are perpendicular to each other—

polarization :-

consider a plane EM. wave proceeding along the positive x-axis with a speed C. The electric field at any instant t is given by

$$E_y = E_{y_0}\left\{\cos\left[\omega\left(t-\frac{x}{c}\right)+\theta\right]\right. \quad [\text{from } y = Y_0 \cos(\omega t+\theta)]$$

Here $t = t-\frac{x}{c}$, where θ is the initial phase angle i.e. at $t=0$, and $x=0$.

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In exponential form, we write-

$$E_y = E_{y_0} \exp\left[i\left\{\omega\left(t-\frac{x}{c}\right)+\theta\right\}\right]$$

As $\left[\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\right]$ and only real part is considered.

Similarly relations also hold good for B_z and H_z

From eqn (13) we have.

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

$$\frac{\partial}{\partial x}\left[E_{y_0}\exp\left[i\left\{\omega\left(t-\frac{x}{c}\right)+\theta\right\}\right]\right]$$

$$= -\frac{\partial}{\partial t}\left[B_{z_0}\exp\left\{i\left(\omega\left(t-\frac{x}{c}\right)+\theta\right)\right]\right]$$

$$E_{y_0}\exp\left[i\left(\omega\left(t-\frac{x}{c}\right)+\theta\right)\right]\left(-\frac{i\omega}{c}\right)$$

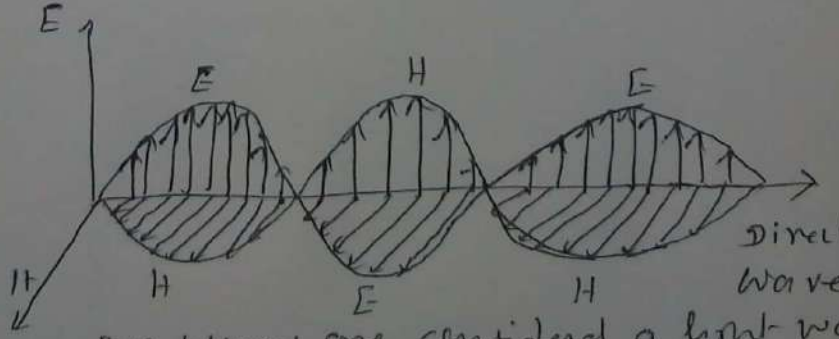
$$= -B_{z_0}\exp\left[i\left[\omega\left(t-\frac{x}{c}\right)+\theta\right)\right](i\omega)$$

$$E_y\left(-\frac{i\omega}{c}\right) = -B_z(i\omega) \quad \text{or} \quad \frac{E_y}{B_z} = c$$

$$\frac{E_y}{H_z} = \mu_0 c = \frac{\mu_0}{\sqrt{\mu_0/\epsilon_0}} = \sqrt{\mu_0/\epsilon_0} = \text{constant}$$

This relation shows that the vector E and H are always in phase with each other, From the above we observed that vector E and vector H are mutually perpendicular each other and propagates in the direction of $E \times H$

Hence if E is parallel to y-axis and H is parallel to z-axis the wave propagates along x-axis as shown in fig.



Em waves are considered a light waves, These Em waves propagates through certain crystals, one vibration are restricted at a particular plane (either E or H) is called polarization of Em waves, It happens when E and H are $\perp^r$ to each other

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