Chapter - 2

Matrices (Continued)

2.1 System of 'm' linear equations in 'n' unknowns

A system of equations in which each unknown appears in first degree only, is called a system of linear equations. A system of m-equations in n-knowns is represented as :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots\ldots\ldots + a_{mn}x_n = b_m \end{aligned} \quad \text{----- (1)}$$

In matrix form the system (1) is represented as :

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots\ldots & a_{mn} \end{bmatrix}_{m\times n}$ is a $m \times n$ coefficient matrix.

$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1}$ is column matrix of n-unknowns and $B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}_{m\times 1}$ is a column matrix formed by the m-constants on right hand side of the system (1).

Solution : A set of values of the n-unknowns $x_1, x_2, \ldots\ldots, x_n$, which satisfy all the m-equations of the system (1) simultaneously is called the **solution** of the system of equations (1).

Consistent system

A system of equations (1) is said to be **consistent** or **compatible** if the system possesses either one or an infinite number of solutions.

Inconsistent system

If the system of equation (1) has no solution then it is called **inconsistent system**.

2.2 System of homogeneous linear equations

If each constant value on the right hand side of the system (1) is equal to zero then (1) is called the system of m-homogeneous equations in n-unknowns. It is given by.

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots\ldots + a_{1n}\, x_n &= 0 \\ a_{21}\, x_1 + a_{22}\, x_2 + \ldots\ldots\ldots + a_{2n}\, x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n &= 0 \end{aligned} \quad \text{---- (2)}$$

In matrix form $A\,X = 0$ where A and X are as defined earlier while $0 = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}_{m\times 1}$ is a $m \times 1$ null matrix.

Trivial solution :

It is straight forward that any system of homogeneous equations possesses one solution namely $x_1 = 0, x_2 = 0, \ldots\ldots, x_n = 0$. This solution is called a **zero solution** or a **trivial solution**.

Consistency of system of homogeneous equations

Since there always exists a trivial solution for any system of homogeneous equations, can conclude that the system of homogeneous equations is always **consistent**.

Non-trivial solution

Any solution of the system of homogeneous equations which is other than a zero or a tri solution is called a non-trivial solution.

i.e., atleast one of the unknowns $x_1, x_2 \ldots\ldots, x_n$ must have a non-zero value.

Theorem : The number of linearly independent solutions of a system $AX = 0$ of m-homogene equations with n-unknowns is $(n - r)$ where r is the rank of A i.e., $\rho(A) = r$.

2.3 Criterion for existence of non-trivial solution

Theorem : The system $AX = 0$ has a non-zero solution if and only if A is singular.

Proof : First suppose that $AX = 0$ has a non-zero solution.

i.e., $\exists$ atleast one linearly independent solution.

$\Rightarrow \quad n - r > 0$ where $r = \rho(A)$ and n is number of columns of A.

$\Rightarrow \quad n > r$

or $\quad r < n$

i.e., $\rho(A) <$ order of the matrix A

$\Rightarrow \quad |A| = 0$

$\Rightarrow \quad A$ is singular.

Conversely suppose that A is singular.

$\Rightarrow \quad |A| = 0$

$\Rightarrow \quad \rho(A) <$ order of A

$\Rightarrow \quad r < n$

$\Rightarrow \quad n > r$

$\Rightarrow \quad n - r > 0$

$\Rightarrow \quad AX = 0$ has atleast one non-zero solution.

Nature of solutions of the system $AX = 0$

If $AX = 0$ is a system of m-homogeneous linear equations in n-unknowns.

$\Rightarrow \quad A$ is of order $m \times n$.

$\Rightarrow \quad \rho(A) = r \leq \min(m, n)$

ase i) When $n < m$: i.e., number of unknowns is less than the number of equations.

$\Rightarrow \quad \rho(A) = r = n$ or $r < n$.

Subcase (a) **When $r = n$** : We know that $AX = 0$ has $n - r = n - n = 0$ non-zero solutions. Hence in this case $AX = 0$ has only trivial solution.

Subcase (b) **When $r < n$** : The system $AX = 0$ has $(n - r)$ linearly independent solutions. Further any linear combination of these $(n - r)$ solution is again a solution $AX = 0$.

Hence $AX = 0$ has infinite number of solutions.

Case ii) When $m < n$: i.e., number of equations is less than number of unknowns.

We know that $\rho(A) = r < m$

$\Rightarrow \quad r < n$

$\Rightarrow \quad n > r$

$\Rightarrow \quad n - r < 0$

Hence there exist infinite number of solutions of $AX = 0$.

Case iii) When $m = n$: i.e., the number equations is equal to the number of unknowns :

In this case A is a square matrix of order n. To have a non-zero solution

$n - r > 0$

$\Rightarrow \quad n > r \Rightarrow r < n$

i.e., $\rho(A) < n$

$\Rightarrow \quad |A| = 0 \qquad (\because \; |A|$ is a minor of order $n)$

$\Rightarrow \quad A$ is singular.

2.4 Solution of a system of homogeneous equations

Working rule :

Step 1 : Write the co-efficient matrix A.

Step 2 : Apply the row-operations (only) to reduce A into the echelon form.

Step 3 : Find $\rho(A) = r$ and check whether $\rho(A) = r < n$ (i.e., rank is less than the number of unknowns) or not.

If $r < n$, conclude that there exist infinite solutions. On the other-hand if $r = n$ conclude that the system has only trivial solution.

Step 4 : If $r < n$, to find the solutions, re-construct the system of equation using the echelon form of A. Clearly this system will have r number of equations.

Step 5 : To find n-unknowns using r-equations assign $(n - r)$ unknowns with arbitrary values. k_1, k_2, etc.

Note : When $m = n$, first ensure $|A| = 0$ and then proceed.

WORKED EXAMPLES

Example : 1

Determine whether the system
$$x+2y+3z=0$$
$$3x+y+2z=0$$
$$2x+3y+z=0$$
possesses a non-trivial solution.

Solution :

Given system has 3-equations in 3-unknowns. This system to have a non-trivial solution the co-efficient matrix A must be singular i.e., $|A|=0$.

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

$\Rightarrow \quad |A| = 1(1-6)-2(3-4)+3(9-2)$

$= 18 \neq 0$ i.e., $|A| \neq 0$

Hence the system has only trivial solution.

Example : 2

Find the value of λ for which the system $x+3y+2z=0;\ x+\lambda y+3z=0;\ x+5y+4z=0$ has a non-zero solution. Hence find the complete solution.

Solution :

Here $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & \lambda & 3 \\ 1 & 5 & 4 \end{bmatrix}$

Since there are three equations and three unknowns, for the existence of non-trivial solution we have

$|A| = 0$

i.e., $|A| = 1(4\lambda-15)-3(4-3)+2(5-\lambda)=0$

$\Rightarrow \quad 4\lambda - 15 - 3 + 10 - 2\lambda = 0$

$\Rightarrow \quad 2\lambda = 8$

$\Rightarrow \quad \lambda = 4$

Thus for $\lambda = 4$, we have $|A|=0$ and hence the given system will have non-trivial solutions.

Now to find the solution : Rewritting A

i.e., $$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & 5 & 4 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to R_3 - 2R_2)$$

This is in echelon form.

$\Rightarrow \quad \rho(A) = r = 2 < 3$, the number of unknowns.

Now reconstructing the system of equations using the final form of A.

$x + 3y + 2z = 0$ ---- (1)

$y + z = 0$ ---- (2)

(2) $\Rightarrow y = -z$ and (1) $\Rightarrow x = -3y - 2z$

$\Rightarrow \quad x = +3z - 2z$

$\Rightarrow \quad x = z$

Now choose $z = k$ arbitrarily so that

$x = k$ and $y = -k$

Hence $x = k$, $y = -k$, $z = k$ represents the complete solution of the given system.

Example : 3

Find the real value of μ for which the system $3x + y + 2z = \mu y$; $2x + 3y + z = \mu z$; $x + 2y + 3z = \mu x$ possesses a non-trivial solutions.

Solution :

Rewritting the given system of equations,

$$3x + (1-\mu)y + 2z = 0$$
$$2x + 3y + (1-\mu)z = 0$$
$$(1-\mu)x + 2y + 3z = 0$$

Since there are three equations in three unknowns, for the given system to have non-trivial solutions, we have,

$$|A| = 0$$

i.e.,
$$\begin{vmatrix} 3 & (1-\mu) & 2 \\ 2 & 3 & (1-\mu) \\ (1-\mu) & 2 & 3 \end{vmatrix} = 0$$

$R_3 \to R_3 + R_2 + R_1$

$$\Rightarrow \begin{vmatrix} 3 & (1-\mu) & 2 \\ 2 & 3 & (1-\mu) \\ (6-\mu) & (6-\mu) & (6-\mu) \end{vmatrix} = 0$$

$$\Rightarrow (6-\mu)\begin{vmatrix} 3 & (1-\mu) & 2 \\ 2 & 3 & (1-\mu) \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (6-\mu)\{3(3-1+\mu)-(1-\mu)(2-1+\mu)+2(2-3)\}=0$$

$$\Rightarrow (6-\mu)\{6+3\mu-1+\mu^2-2\}=0$$

$$\Rightarrow (6-\mu)(\mu^2+3\mu+3)=0$$

$$\Rightarrow 6-\mu=0, \quad \mu^2+3\mu+3=0$$

$$\Rightarrow \mu=6, \qquad \mu=\frac{-3\pm\sqrt{9-12}}{2}=\frac{-3\pm i\sqrt{3}}{2}$$

Hence the real value of $\mu = 6$, for which the given system has a non-zero solution

---

Example : 4

Solve the system of equations $x-y+2z=0$; $4x+y+5z=0$; $3x+2y+z=0$.

Solution :

Here

$$A=\begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -3 \\ 0 & 5 & -5 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & -2 \end{bmatrix} \quad (R_3 \to R_3 - R_2)$$

i.e., The number of non-zero rows is 3.

$\Rightarrow \quad \rho(A) = 3 =$ number of unknowns.

$\Rightarrow$ The given system has only trivial solution.

Example : 5

Solve the system $x+17y+4z=0;\ 2x-y-3z=0;\ -3x+5y-4z=0\ ;\ x+y+z=0.$

Solution :

$$\text{Here} \quad A = \begin{bmatrix} 1 & 17 & 4 \\ 2 & -1 & -3 \\ -3 & 5 & -4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ -3 & 5 & -4 \\ 1 & 17 & 4 \end{bmatrix} \quad (R_1 \leftrightarrow R_4)$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 8 & -1 \\ 0 & 16 & 3 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 - R_1 \end{pmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & -43 \\ 0 & 0 & -71 \end{bmatrix} \quad \begin{pmatrix} R_3 \to 3R_3 + 8R_2 \\ R_4 \to 3R_4 + 16R_2 \end{pmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & -43 \\ 0 & 0 & 0 \end{bmatrix} \quad (R_4 \to 43R_4 - 71R_3)$$

i.e., The number of non-zero rows is 3.

$\Rightarrow \quad \rho(A) = 3 =$ number of unknowns.

$\Rightarrow$ There exist no non-trivial solutions.

Hence the system has only zero-solutions.

Example : 6

$-x+y+z=0$; $3x+y-z=0$; $2x+2y=0$

Solution :

Here $\quad A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

$\Rightarrow \quad A \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 4 & 2 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 2R_1 \end{pmatrix}$

$\Rightarrow \quad A \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to R_3 - R_2)$

Since there are 2 non-zero rows

$\rho(A) = 2 < 3$, the number of unknowns.

$\therefore$ The given system possesses a non-trivial solution.

Reconstructing the system from final form of A, We get

$-x + y + z = 0$ ---- (1)

$4y + 2z = 0$ ---- (2)

$(2) \Rightarrow 2y + z = 0 \Rightarrow y = -\dfrac{z}{2}$

$\therefore \quad (1) \Rightarrow -x - \dfrac{z}{2} + z = 0 \Rightarrow x = \dfrac{z}{2}$

Choose $z = k$ arbitrarily, so that,

$y = \dfrac{-k}{2}$ and $x = \dfrac{k}{2}$

Hence $x = \dfrac{k}{2}$, $y = \dfrac{-k}{2}$, $z = k$ represents the required solution.

***Example* : 7**

Solve $2x_1 + x_2 + x_4 = 0$; $8x_1 + 4x_2 + 2x_3 + 6x_4 = 0$; $6x_1 + 3x_2 + 4x_3 + 7x_4 = 0$.

Solution :

Here the unknowns are x_1, x_2, x_3 and x_4.

The coefficient matrix is

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 8 & 4 & 2 & 6 \\ 6 & 3 & 4 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (R_3 \to R_3 - 2R_2)$$

Since there are two non-zero rows,

$\rho(A) = 2 < 4$, the number of unknowns.

$\Rightarrow$ The system has non-trivial solutions.

Reconstructing the system from the final form of A.

$\Rightarrow \quad 2x_1 + x_2 + x_4 = 0$ ---- (1)

$2x_3 + 2x_4 = 0$ ---- (2)

(2) $\Rightarrow x_3 = -x_4$ and (1) $\Rightarrow x_1 = \dfrac{-1}{2}(x_2 + x_4)$

Choose $x_2 = k_1$ and $x_4 = k_2$ arbitrarily so that,

$$x_3 = -k_2 \text{ and } x_1 = -\frac{1}{2}(k_1 + k_2)$$

Hence $x_1 = \dfrac{-1}{2}(k_1 + k_2)$; $x_2 = k_1$; $x_3 = -k_2$, $x_4 = k_2$ is the required solution.

***Example* : 8**

Solve

$$4x + 6y - 2z - 2u = 0$$
$$-2x + 3y + z - u = 0$$
$$3x - 4y - z + 2u = 0$$

Solution :

Here

$$A = \begin{bmatrix} 4 & 6 & -2 & -2 \\ -2 & 3 & 1 & -1 \\ 3 & -4 & -1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 6 & -2 & -2 \\ 0 & 12 & 0 & -4 \\ 0 & -34 & 2 & 14 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to 2R_2 \ldots \\ R_3 \to 4R_3 - \ldots \end{pmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & -17 & 1 & 7 \end{bmatrix} \qquad \begin{pmatrix} R_1 \to \frac{1}{2} \ldots \\ R_2 \to \frac{1}{4} \ldots \\ R_3 \to \frac{1}{2} \ldots \end{pmatrix}$$

i.e., $$A \sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \qquad (R_3 \to 3R_3 + 17 \ldots$$

Since there are 3 non-zero rows.

$\rho(A) = 3 < 4$, the number of unknowns.

Reconstructing the system using the final form of A.

$$\Rightarrow \quad 2x + 3y - z - u = 0 \quad \text{—(1)}$$
$$3y - u = 0 \quad \text{—(2)}$$
$$3z - 4u = 0 \quad \text{—(3)}$$

(3) $\Rightarrow z = \frac{4}{3}u$; (2) $\Rightarrow y = \frac{1}{3}u$; (1) $\Rightarrow 2x + u - \frac{4u}{3} - u = 0$.

$$\Rightarrow x = \frac{2u}{3}.$$

Choose $u = k$ arbitrarity so that,

$x = \frac{2k}{3}, y = \frac{k}{3}, z = \frac{4k}{3}, u = k$.

This is the required complete solution.

Example : 9

Solve $-w+x-2y+z=0$; $3w+x+z=0$, $w+x-y+z=0$.

Solution :

$$A = \begin{bmatrix} -1 & 1 & -2 & 1 \\ 3 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & 4 & -6 & 4 \\ 0 & 2 & -3 & 2 \end{bmatrix} \qquad (R_2 \to R_2 + 3R \quad R_3 \to R_3 + R_1)$$

$$\sim \begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & 4 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to 2R_3 - R_2$$

$\Rightarrow \rho(A) = 2 < 4$, the number of unknowns.

Reconstructing the system using the final form of A.

$-w+x-2y+z=0$ ---- (1

$4x-6y+4z=0$ ---- (2

$(2) \Rightarrow x = \frac{3}{2}y - z$; $(1) \Rightarrow w = \frac{3}{2}y - z - 2y + z$

$$\Rightarrow w = -\frac{1}{2}y$$

Choose $y = k_1$, and $z = k_2$ arbitrarily so that, $w = -\frac{1}{2}k_1$; $x = \frac{3}{2}k_1 - k_2$; $y = k_1$, $z = k$ constitutes the required general solution.

Example : 10

Solve $-x_1+3x_2-7x_3-6x_4=0$; $6x_1-4x_2+6x_3+8x_4=0$;

$4x_1-x_2+x_3+x_4=0$; $-2x_1+2x_2-5x_3-3x_4=0$.

Solution :

Here
$$A = \begin{bmatrix} -1 & 3 & -7 & -6 \\ 6 & -4 & 6 & 8 \\ 4 & -1 & 1 & 1 \\ -2 & 2 & -5 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 3 & -7 & -6 \\ 0 & 14 & -36 & -28 \\ 0 & 11 & -27 & -23 \\ 0 & -4 & 9 & 9 \end{bmatrix} \quad (R_2 \to R_2 + 6R_1 ; R_3 \to R_3 + \ldots \; R_4 \to R_4 - \ldots$$

$$\sim \begin{bmatrix} -1 & 3 & -7 & -6 \\ 0 & 7 & -18 & -14 \\ 0 & 11 & -27 & -23 \\ 0 & -4 & 9 & 9 \end{bmatrix} \quad \left(R_2 \to \frac{1}{2} \ldots\right.$$

$$\sim \begin{bmatrix} -1 & 3 & -7 & -6 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \\ 0 & -4 & 9 & 9 \end{bmatrix} \quad (R_3 \to R_3 - \ldots$$

$$\sim \begin{bmatrix} -1 & 3 & -7 & -6 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (R_4 \to R_4 + \ldots$$

$$A \sim \begin{bmatrix} -1 & 3 & -7 & -6 \\ 0 & 7 & -18 & -14 \\ 0 & 0 & 9 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (R_3 \to 7R_3 - 4R_2$$

This is in the echelon form and has 3 non-zero rows.

⇒ $\rho(A) = 3 < 4$, the number of unknowns,

⇒ The system has non-trivial solutions.

Now reconstructing the system from the final form of A.

⇒ $-x_1 + 3x_2 - 7x_3 - 6x_4 = 0$ ---- (1)

$7x_2 - 18x_3 - 14x_4 = 0$ ---- (2)

$9x_3 - 7x_4 = 0$ ---- (3)

(3) ⇒ $x_3 = \dfrac{7}{9} x_4$

$(2) \Rightarrow x_2 = \frac{1}{7}\left\{18\left(\frac{7}{9}\right)x_4 + 14x_4\right\} \Rightarrow x_2 = 4x_4$

$(1) \Rightarrow x_1 = 3(4x_4) - 7\left(\frac{7}{9}\right)x_4 - 6x_4 \Rightarrow x_1 = \left(\frac{108-49-54}{9}\right)x_4$

$$\Rightarrow x_1 = \frac{5}{9}x_4$$

On choosing $x_4 = k$ arbitrarily, we have,

$$x_1 = \frac{5}{9}k,\ x_2 = 4k,\ x_3 = \frac{7}{9}k \text{ and } x_4 = k.$$

This is the required complete solution.

Exercise

1. Determine whether the system $3x+2y+z=0$, $x-y+2z=0$, $4x+y+5z=0$ possesses a non-trivial solution.
2. Find λ for which the system $\lambda x+5y+2z=0$, $x+y+z=0$, $2x+3y=0$ has a non-zero solution. (Ans : $\lambda = 4$)
3. Find λ for which $\lambda x+2y-z=0$; $3x+y-z=0$; $2x-y+2z=0$ possesses a non-trivial solution. (Ans : $\lambda = 11$)
4. Determine whether the system $3x-9y+6z=0$; $7x-21y+14z=0$; $-x+3y-2z=0$ has a non-trivial solution.
5. Find λ for which $x+3y+2z=0$; $x+2y+z=0$; $7x+4y=-\lambda z$ has a non-zero solution (Ans : $\lambda = 3$)
6. Find the real values of k for which the system

   $8x-6y+2z=kx$;

   $6x-7y+4z=ky$;

   $2x-4y+3z=kz$, has a non-zero solution (Ans : $k = 0, 3, 5$)
7. Find the real value of k such that the system $x-2y+kz=0$; $2x-(3+k)y+2z=0$; $(2-k)x-2y+z=0$ has a non-zero solution. (Ans : $k=-3$)
8. Solve $-x+4y-5z=0$; $6x+4y+2z=0$; $2x-y+3z=0$. (Ans : $x=-k, y=k, z=k$)
9. Solve $6x+4y+2z=0$; $2x+4y+6z=0$; $2x+3z=0$; $y+5z=0$. (Ans : No non-trivial solution)

10. Solve $-2x_1+4x_2+x_3-2x_4=0$

$2x_1-3x_2-x_3+x_4=0$

$4x_1+6x_2-2x_3-2x_4=0$

(Ans : $x_1=\frac{k_1}{3}$; $x_2=2k_2$; $x_3=k_1$; $x_4=k_2$)

2.5 System of non-homogeneous linear equations

If atleast one of the values on right-hand side of the system of equations

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\dots\dots\dots+a_{1n}x_n&=b_1\\ a_{21}x_1+a_{22}x_2+\dots\dots\dots+a_{2n}x_n&=b_2\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots&\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots&\\ a_{m1}x_1+a_{m2}x_2+\dots\dots\dots+a_{mn}x_n&=b_m \end{aligned} \qquad \text{-----(1)}$$

is non-zero (i.e., each of $b_1, b_2, \dots\dots, b_m$ are not simultaneously zero) then the system (1) is called a system of non-homogeneous linear equation.

In the matrix form, system (1) can be represented as $AX=B$.

Here B is a non-zero column matrix formed by the constant values present on the rhs of (1).

2.6 Consistency criterion

To check the consistency of the given system $AX=B$, we have to first form the **augmented matrix**.

$$[A:B]=\begin{bmatrix} a_{11}\ a_{12}\ \dots\dots\dots\ a_{1n} : b_1\\ a_{21}\ a_{22}\ \dots\dots\dots\ a_{2n} : b_2\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ a_{m1}\ a_{m2}\dots\dots\dots a_{mn} : b_m \end{bmatrix}$$

by combining the coefficient matrix A and the column matrix B (seperated by:)

Theorem : The necessary and sufficient condition for the system $AX=B$ to be consistent (i.e., possess a solution) is that $\rho(A)=\rho[A:B]=r$ (say)

i.e., A and $[A:B]$ must be of the same rank r.

•

2.7 Criterion for the uniqueness of the solution

Case i) If $m = n$: (i.e., the number of equations is equal to the number of unknowns, so that A is a square matrix).

Then the system $AX = B$ has a unique solution if A is non-singular i.e., $|A| \neq 0$.

i.e., $\rho(A) = \rho[A:B] = r = n$.

On the other hand if A is singular

i.e., $|A| = 0 \Rightarrow \rho(A) = \rho[A:B] = r < n$

then the system $AX = B$ has infinite solutions.

Case ii) When $m > n$: The system has unique solution if $\rho[A:B] = \rho(A) = r = n$.

On the otherhand if $\rho[A:B] = \rho(A) = r < n$, the system will have infinite solutions.

Case iii) When $m < n$: The system will have infinite solutions ($\because$ in this case $\rho(A) = r < n$).

Working rule

Step 1) : Form the augmented matrix $[A:B]$.

Step 2) : Apply row-operations (only) to reduce $[A:B]$ into the echelon form.

Step 3) : Count the number of non-zero rows in $[A:B]$ and also in A (by hiding the last column of $[A:B]$ to get $\rho[A:B]$ and $\rho(A)$.

Step 4) : If $\rho[A:B] \neq \rho(A)$ conclude that the system is in-consistent.

If $\rho[A:B] = \rho(A) = n$ conclude that the system is consistent and has a unique solution.

If $\rho[A:B] = \rho(A) < n$ conclude that the system is consistent and has infinitely many solutions.

Step 5) : Reconstruct the system of equations using the final form of $[A:B]$ and solve the same.

WORKED EXAMPLES

Example : 1

Verify the system $x + y - z = 1$

$4x + 4y - z = 2$

$6x + 6y + 2z = 3$, for consistency and hence solve.

Solution :

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 4 & 4 & -1 & : & 2 \\ 6 & 6 & 2 & : & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 0 & 0 & 3 & : & -2 \\ 0 & 0 & 8 & : & -3 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 6R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 0 & 0 & 3 & : & -2 \\ 0 & 0 & 0 & : & 7 \end{bmatrix} \qquad (R_3 \to 3R_3 - 8R_2)$$

$\Rightarrow \rho(A:B) = 3$ and $\rho(A) = 2$

i.e., $\rho(A:B) \neq \rho(A)$

Hence the given system is inconsistent.

Example : 2

Verify the system $2x + 4y + 4z = 2$

$3x + 2y + 2z = 3$

$2x + y + z = 2$, for consistency and hence solve.

Solution :

The augmented matrix is

$$[A:B] = \begin{bmatrix} 2 & 4 & 4 & : & 2 \\ 3 & 2 & 2 & : & 3 \\ 2 & 1 & 1 & : & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & 4 & : & 2 \\ 0 & -8 & -8 & : & 0 \\ 0 & -3 & -3 & : & 0 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to 2R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & 4 & : & 2 \\ 0 & 1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to -\frac{1}{8} R_2 \\ R_3 \to -\frac{1}{3} R_2 \end{pmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & 4 & : & 2 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad (R_3 \to R_3 - R_2)$$

$\Rightarrow \quad \rho(A) = 2 \; ; \; \rho[A:B] = 2$

i.e., $\rho(A) = \rho[A:B] = 2 < 3$, the number of unknowns

$\Rightarrow$ The given system is consistent and has infinite solutions. Now reconstructing the system of equations from the final form of $[A:B]$.

$\Rightarrow \quad 2x + 4y + 4z = 2$ ---- (1)

$y + z = 0$ ---- (2)

(2) $\Rightarrow y = -z$; (1) $\Rightarrow 2x = 2 - 4y - 4z$

$\Rightarrow \quad x = 1 - 2y - 2z$

$\Rightarrow \quad x = 1 - 2(-z) - 2z$

$\Rightarrow \quad x = 1$

On choosing $z = k$ arbitrarily we have $x = 1, y = -k, z = k$ is the required solution.

Example : 3

Check the consistency of $x - y - z = 3$; $-x - 10y + 3z = -5$; $2x - y + 2z = 2$ and hence solve.

Solution :

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & -1 & -1 & : & 3 \\ -1 & -10 & 3 & : & -5 \\ 2 & -1 & 2 & : & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & : & 3 \\ 0 & -11 & 2 & : & -2 \\ 0 & 1 & 4 & : & -4 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & : & 3 \\ 0 & -11 & 2 & : & -2 \\ 0 & 0 & 46 & : & -46 \end{bmatrix} \qquad (R_3 \to 11R_3 + R_2)$$

$\Rightarrow \rho[A:B] = \rho(A) = 3$ = number of unknowns.

Hence the given system is consistent and has **unique solution.**

On reconstructing the system from final form of A, we have

$x - y - z = +3$ ---- (1)

$-11y + 2z = -2$ ---- (2)

$46z = -46$ ---- (3)

(3) $\Rightarrow z = \dfrac{-46}{46} \Rightarrow z = -1.$

(2) $\Rightarrow y = -\dfrac{1}{11}(-2 - 2z) \Rightarrow y = 0.$

(1) $\Rightarrow x = 3 + y + z$

$\Rightarrow \quad x = 2$

Hence $x = 2, y = 0, z = -1$ is the required solution.

ample : 4

Solve $3x_1 + x_2 + 2x_3 = 3$; $2x_1 - 3x_2 - x_3 = -3$; $x_1 + 2x_2 + x_3 = 4$.

ution :

The augmented matrix is

$$[A:B] = \begin{bmatrix} 3 & 1 & 2 & : & 3 \\ 2 & -3 & -1 & : & -3 \\ 1 & 2 & 1 & : & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -3 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix} \qquad (R_1 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & 0 & 8 & : & -8 \end{bmatrix} \qquad (R_3 \to 7R_3 - 5R_2)$$

$\Rightarrow \quad \rho(A) = \rho\left[A : B\right] = 3$ = number of unknowns.

Hence the system is consistent and has unique solution.

Reconstructing the system from final form of $[A : B]$, we have

$$x_1 + 2x_2 + x_3 = 4 \quad \text{---- (1)}$$
$$-7x_2 - 3x_3 = -11 \quad \text{---- (2)}$$
$$8x_3 = -8 \quad \text{---- (3)}$$

$(3) \Rightarrow x_3 = -1\,; (2) \Rightarrow x_2 = \frac{1}{7}(11 - 3x_3) \Rightarrow x_2 = 2$

$(1) \Rightarrow x_1 = 4 - 2x_2 - x_3 \Rightarrow x_1 = 1$

Hence $x_1 = 1, x_2 = 2, x_3 = -1$ is the required solution.

ample : 5

Solve $x + 2y - 3z = -4\,; 2x + 3y + 2z = 2\,; 3x - 3y - 4z = 11$.

lution :

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 2 & -3 & : & -4 \\ 2 & 3 & 2 & : & 2 \\ 3 & -3 & -4 & : & 11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & -4 \\ 0 & -1 & 8 & : & 10 \\ 0 & -9 & 5 & : & 23 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & -4 \\ 0 & -1 & 8 & : & 10 \\ 0 & 0 & -67 & : & -67 \end{bmatrix} \qquad (R_3 \to R_3 - 9R_2)$$

$\Rightarrow \rho\left[A : B\right] = \rho(A) = 3$ = number of unknowns.

$\therefore$ The system is consistent and has unique solution.

Reconstructing the system from the final form of $[A : B]$.

$$\Rightarrow \quad x + 2y - 3z = -4 \quad \text{---- (1)}$$
$$-y + 8z = 10 \quad \text{---- (2)}$$
$$-67z = -67 \quad \text{---- (3)}$$

$\therefore$ $(3) \Rightarrow z = 1\,; (2) \Rightarrow y = 8z - 10 \Rightarrow y = -2\,; (3) \Rightarrow x = -4 - 2y + 3z \Rightarrow x = 3$

Hence $x = 3, y = -2, z = 1$ is the requird solution.

Example : 6

Solve $x_1 + x_2 - x_3 = 1$; $x_1 - x_2 - x_3 = -1$; $3x_1 + x_2 - 2x_3 = 3$.

Solution :

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 1 & -1 & -1 & : & -1 \\ 3 & 1 & -2 & : & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 0 & -2 & 0 & : & -2 \\ 0 & -2 & 1 & : & 0 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 2 \end{bmatrix} \quad \left(R_2 \to -\frac{1}{2} R_2 \; ; \; R_3 \to R_3 - R_2 \right)$$

$\Rightarrow$ $\rho[A : B] = \rho(A) = 3$ = number of unknowns

$\Rightarrow$ The system is consistent and has unique solutions.

Reconstructing the system from the final form of $[A : B]$

$\Rightarrow$ $x_1 + x_2 - x_3 = 1$ ---- (1)

$x_2 = 1$

$x_3 = 2$

$\therefore$ (1) $\Rightarrow x_1 = 1 - x_2 + x_3 \Rightarrow x_1 = 2$

Hence $x_1 = 2, x_2 = 1, x_3 = 2$ is the required solution.

Example : 7

Solve $x_1 + x_2 + x_3 = 1$; $x_1 + 2x_2 + 3x_3 = 4$; $2x_1 + 6x_2 + 10x_3 = 14$; $x_1 + 4x_2 + 7x_3 = 10$.

Solution :

Here

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 3 & : & 4 \\ 2 & 6 & 10 & : & 14 \\ 1 & 4 & 7 & : & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 2 & : & 3 \\ 0 & 4 & 8 & : & 12 \\ 0 & 3 & 6 & : & 9 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad \begin{pmatrix} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 3R_2 \end{pmatrix}$$

$\Rightarrow \rho[A:B] = \rho(A) = 2 < 3$, the number of unknowns.

$\Rightarrow$ The system is consistent and has a infinite solutions.

Reconstructing the system of equations from the final form of $[A : B]$,

$x_1 + x_2 + x_3 = 1$ ---- (1)

$x_2 + 2x_3 = 3$ ---- (2)

$(2) \Rightarrow x_2 = 3 - 2x_3$

$(1) \Rightarrow x_1 = 1 - x_2 - x_3 = 1 - 3 + 2x_3 - x_3 = x_3 - 2.$

Choosing $x_3 = k$ arbitrarily so that

$$x_1 = k - 2\,;\; x_2 = 3 - 2k,\; x_3 = k$$

This is the required solution.

Example : 8

Find the value of λ for which the system $x + y + z = 1;\; x + 2y + 4z = \lambda;\; x + 4y + 10z = \lambda^2$ is consistent and hence solve.

Solution :

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & \lambda \\ 1 & 4 & 10 & : & \lambda^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda - 1 \\ 0 & 3 & 9 & : & \lambda^2 - 1 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda-1 \\ 0 & 0 & 0 & : & \lambda^2-3\lambda+2 \end{bmatrix} \qquad (R_3 \to R_3 - 3R_2)$$

Clearly $\rho(A) = 2$.

For the system to be consistent $\rho[A:B] = \rho(A)$

i.e., $\rho[A:B] = 2$

Thus the third row of $[A : B]$ must be zero

$\Rightarrow \quad \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \quad (\lambda-1)(\lambda-2) = 0$

$\Rightarrow \quad \lambda = 1, 2$

Hence for $\lambda = 1, 2$, the above system is consistent and since $\rho[A:B] = \rho(A) = 2 < 3$, number of unknowns, the system has unique solution.

Case 1 : When $\lambda = 1$, (1) $\Rightarrow$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Reconstructing the system

$\Rightarrow \quad x + y + z = 1$

$y + 3z = 0 \Rightarrow y = -3z.$ ---- (2

$\therefore$ (1) $\Rightarrow x = 1 - y - z = 1 + 3z - z = 1 + 2z.$

Choosing $z = k_1$ arbitrarily, so that, $x = 1 + 2k_1,\ y = -3k_1,\ z = k_1$

This is the required general solution.

Case 2 : When $\lambda = 2$, (1) $\Rightarrow$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Reconstructing the system

$\Rightarrow \quad x + y + z = 1$

$y + 3z = 1 \Rightarrow y = 1 - 3z.$ ---- (3)

$\therefore$ (3) $\Rightarrow x = 1 - y - z = 1 - 1 + 3z - z = 2z.$

Choosing $z = k_2$ arbitrarily, so that, $x = 2k_2,\ y = 1 - 3k_2,\ z = k_2$

This is the required complete solution.

Example : 9

Find for what values of λ and μ the system

$$x+y+z=6$$
$$2x+4y+6z=20$$
$$x+2y+\lambda z=\mu$$

has (i) a unique solution (ii) infinite solutions (iii) no solution.

Solution :

The augmented matrix is

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 2 & 4 & 6 & : & 20\\ 1 & 2 & \lambda & : & \mu\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 2 & 4 & : & 8\\ 0 & 1 & \lambda-1 & : & \mu-6\end{bmatrix} \quad (R_2 \to R_2 - 2 \quad R_3 \to R_3 - R_1)$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 1 & \lambda-1 & : & \mu-6\end{bmatrix} \quad \left(R_2 \to \frac{1}{2} R_2\right.$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 0 & \lambda-3 & : & \mu-10\end{bmatrix} \quad (R_3 \to R_3 - R_2$$

(i) Unique solution : This is possible only when $\rho[A:B]=\rho(A)=3=$ the number of unknown

i.e., There must be three non-zero rows in both A and $[A:B]$.

$\Rightarrow \lambda-3\neq 0 \Rightarrow \lambda\neq 3$ and whatever may be the value of μ.

(ii) Infinite solution : This is possible only when $\rho[A:B]=\rho(A)=2<3$, the number o unknowns.

There must be 2 non-zero rows in both A and $[A:B]$.

$\Rightarrow \quad \lambda-3=0$ and $\mu-10=10$.

$\Rightarrow \quad \lambda=3$ and $\mu=10$.

(iii) No solution : This system is inconsistent only when $\rho[A:B]\neq\rho(A)$.

$\therefore$ We must have $\rho(A)=2$ and $\rho(A:B)=3$.

$\Rightarrow \quad \lambda-3=0$ and $\mu-10\neq 0$.

$\Rightarrow \quad \lambda=3$ and $\mu\neq 10$.

Example : 10

Show that the system $x-2y+z=l$, $x+y-2z=m$; $-2x+y+z=n$ has the solutio... $l+m+n=0$. Solve if $l=1$ $m=-2$, $n=1$.

Solution :

$$[A:B] = \begin{bmatrix} 1 & -2 & 1 & : & l \\ 1 & 1 & -2 & : & m \\ -2 & 1 & 1 & : & n \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & : & l \\ 0 & 3 & -3 & : & m-l \\ 0 & -3 & 3 & : & n+2l \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - \\ R_3 \to R_3 + 2 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & : & l \\ 0 & 3 & -3 & : & m-l \\ 0 & 0 & 0 & : & l+m+n \end{bmatrix} \quad (R_3 \to R_3 +$$

For the given system to have solution

$$\rho(A) = \rho[A:B]$$

Since there are two non-zero rows in A, so must be in $[A:B]$

$\therefore \quad l+m+n=0.$

This is the required condition for

$\rho(A) = \rho(A:B) = 2 < 3$, the number of unknowns and hence the system possess t... solution.

Now if $l=1$, $m=-2$, $n=1$, we have

$$[A:B] = \begin{bmatrix} 1 & -2 & 1 & : & 1 \\ 0 & 3 & -3 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & : & 1 \\ 0 & 1 & -1 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad \left(R_3 \to \frac{1}{3} R_3\right.$$

Reconstructing the system from the final form of $[A:B]$

$\Rightarrow \quad x-2y+z=1$

$y-z=-1 \Rightarrow y=z-1.$ ---- (1

$\therefore$ (1) $\Rightarrow x = 1 + 2y - z = 1 + 2z - 2 - z \Rightarrow x = z - 1$.

Choose $z = k$ arbitrarily so that

$$x = k - 1, y = k - 1, z = k$$

This is the required solution.

Exercise

1. Verify the system $2x - y - z = -4$ for consistency. (Ans : Inconsistent)
$$-x + 2y - z = 2$$
$$x + y - 2z = 5$$

2. Verify the system $-x + y - 2z = -2$ for consistency and hence solve.
$$2x + 2y + 2z = 8$$
$$2x + y - z = 1$$
(Ans : $x = \frac{3}{7}, y = \frac{13}{7}, z = \frac{12}{7}$)

3. Solve $x + y - z = -4$; $x - 2y + 3z = 5$; $4x + 3y + 4z = 7$. (Ans : $x = 2, y = 1, z = 3$)

4. Check for consistency and solve, $x + 2y + 3z = 14$, $3x + y + 2z = 11$, $2x + 3y + z = 11$. (Ans : $x = 1, y = 2, z = 3$)

5. Check for consistency and solve, $x_1 + 2x_2 + x_3 = 2$; $2x_1 + 4x_2 + 3x_3 = 3$; $3x_1 + 6x_2 + 5x_3 = 14$. (Ans : In consistent)

6. Solve $x + y = 0$; $y + z = 1$; $z + x = 3$. (Ans : $x = 1, y = -1, z = 2$)

7. Solve $x + 3y + 4z = 8$; $2x + y + 2z = 5$; $5x + y + z = 7$. (Ans : $x = y = z = 1$)

8. Solve $2x + 3y + 3z = 5$; $x - 2y - z = -4$; $3x - y - 2z = 3$. (Ans : $x = \frac{2}{3}, y = \frac{31}{9}, z = \frac{-20}{9}$)

9. Solve $x - y - z = 1$; $2x + y + z = 2$; $x - 2y + z = 4$. (Ans : $x = 1, y = -1, z = 1$)

0. Solve $x + 2y + z = 7$; $x + 3z = 11$; $2x - 3y = 1$. (Ans : $x = 2, y = 1, z = 3$)

1. Solve $x + 3y + 6z = 2$; $3x - y + 4z = 9$; $x - 4y + 2z = 7$. (Ans : $x = 2, y = -1, z = \frac{1}{2}$)

2. Solve $x + y + z = 1$; $x + 2y + 3z = 6$; $x + 3y + 4z = 6$. (Ans : $x = 1, y = -5$, $z = 5$)

13. Solve $x_1+x_2+x_3=4$; $x_1+2x_2+3x_3=4$; $x_1+4x_2+9x_3=6$.

14. Solve $x_1+x_2+x_3=4$; $2x_1-x_2-3x_3=2$; $x_1+2x_2-x_3=9$.

(Ans : $x_1=2, x_2=3, x_3=$

15. Solve $x+y-z=10$; $5x+2y+3z=-2$; $2x+3y+z=9$. (Ans : $x=1, y=4, z=$

16. Solve $2x+4y-2z=6$; $6x-2y+4z=2$; $2x-2y+3z=2$. (Ans : $x=-1, y=4, z=$

17. Solve $-x-2y+z=-3$; $3x-y+2z=1$; $4x-4y+6z=4$. (Ans : $x=-1, y=4, z=$

18. Solve $x_1-3x_2-8x_3=-10$; $3x_1+x_2-4x_3=0$; $2x_1+5x_2+6x_3=13$.

(Ans : $x=2k-1, y=-2k+3, z=$

19. Solve $-x_1+x_2-x_3=1$; $4x_1-4x_2+6x_3=4$; $3x_1-x_2+2x_3=1$ $x_1+2x_2-5x_3=-13$.

(Ans : $x=-1, y=z=$

20. Solve $x+y+z=4$; $4x+2y+2z=4$; $x-z=1$;$2y-z=0, -y+z=3$.

(Ans : Inconsiste

21. Solve $x+y+z=6$; $x-y+z=2$; $2x-y+3z=9$. (Ans : $x=1, y=2, z=$

22. Solve $2x+4y+6z=4$; $2x+4y+5z=3$; $3x+5y+6z=4$ (Ans : $x=1, y=-1, z=$

23. Find λ for which the system $\lambda x+2y-2z=1$; $4x+2\lambda y-z=2$; $6x+6y+\lambda z=3$ has a uniq solution. (Ans : $\lambda \neq 2$

24. Find λ and μ for which the system $x+y+z=3$ has

$$x+2y+2z=6$$
$$x+\lambda y+3z=\mu$$

(i) unique solution (ii) infinite solution (iii) no solutions.

25. Find λ and μ for which the system $x+2y+z=3$ has

$$2x+6y+8z=10$$
$$x+3y+\lambda z=\mu$$

(i) unique solution (ii) infinite solution (iii) no solutions.

26. Find λ and μ such that the system $-x+2y-3z=-1$; $3x+y-z=4$; $2x-2y+\lambda z=\mu$ has
(i) unique solution (ii) infinite solutions (iii) no solutions.

Ans : (i) $\lambda \neq \frac{22}{7}$ and whatever may be the value of

(ii) $\lambda=\frac{22}{7}, \mu=\frac{16}{7}$; (iii) $\lambda=\frac{22}{7}, \mu \neq \frac{16}{7}$

27. Find the condition for which the system $x_1+2x_2+3x_3=l$, $5x_1+8x_2+x_3=$ $2x_1+3x_2-x_3=n$ is consistent

(Ans : $l+2n=m$

2.8 Eigen values and Eigen vectors

(This section is not included in GUK Syllabus)

If A is a square matrix of order $n\times n$, then $A-\lambda I$ is also a square matrix of order $n\times n$. Here I is identity matrix of order $n\times n$ & λ is the unknown constant. Further, $|A-\lambda I|$ is a determinant, which on expansion gives a polynomial of degree n in λ and is called a characteristic polynomial of A.

Definition: The equation $|A-\lambda I|=0$ is called a **characteristic equation** of a matrix A. This is a polynomial equation of degree n with the unknow λ. On solving the characteristic equation we get n-values for λ. These are called as the **eigen values** or **characteristic roots or latent roots** of a square matrix A.

Corresponding to each eigen value λ_i of A there exists a non-zero vector X such that $(A-\lambda_i I)X=0$ or $AX=\lambda_i X$.

Such a vector X is called as the **eigen vector** or **characteristic vector** or **latent vector** of A corresponding to λ_i.

Working rule to find eigen values & eigen vectors:

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ is a given square matrix

Step (1) Form the characteristic equation

$$|A-\lambda I|=0$$

i.e.,
$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13}\\ a_{21} & a_{22}-\lambda & a_{23}\\ a_{31} & a_{32} & a_{33}-\lambda\end{vmatrix}=0.$$

This on expansion becomes $b_0\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0$ where b_i 's are the numeric constants.

Step (2) Solve the above cubic equation to get the eigen values $\lambda=\lambda_1, \lambda_2$ & λ_3.

Step (3) To find the eigen vector $X=(x_1, x_2, x_3)$ corresponding to $\lambda=\lambda_1$ form the system of equations $(A-\lambda_1 I)X=0$

$$\begin{bmatrix} a_{11}-\lambda_1 & a_{12} & a_{13}\\ a_{21} & a_{22}-\lambda_1 & a_{33}\\ a_{31} & a_{32} & a_{33}-\lambda_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

or
$$\begin{aligned}(a_{11}-\lambda_1)x_1+a_{12}x_2+a_{13}x_3 &= 0\\ a_{21}x_1+(a_{22}-\lambda_1)x_2+a_{23}x_3 &= 0\\ a_{31}x_1+a_{32}x_2+(a_{33}-\lambda_1)x_3 &= 0\end{aligned}$$

Step (4) Solve the above system to get x_1, x_2 & x_3. Then $X=(x_1, x_2, x_3)$ is the required eigen vector corresponding to λ_1.

Step (5) Repeate the steps (3) & (4) for $\lambda=\lambda_2$ & λ_3 to find the eigen vectors corresponding to λ_2 & λ_3.

WORKED EXAMPLES

Example : 1

Find the eigen values & eigen vectors of $\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$

Solution :

Let

$$A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) - 12 = 0$$

$\Rightarrow \; 6 - 5\lambda + \lambda^2 - 12 = 0$

$\Rightarrow \; \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow \; (\lambda - 6)(\lambda + 1) = 0$

$\Rightarrow \; \lambda = 6, -1$ are the required eigen values.

To find eigen vector corresponding to $\lambda = 6$:

Let $X = (x_1, x_2)$, be the required eigen vector

Consider $\quad (A - \lambda I)X = 0$

with $\lambda = 6$

$$\Rightarrow \begin{bmatrix} 3-6 & 4 \\ 3 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ (using (

$$\Rightarrow \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3x_1 + 4x_2 \\ 3x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} -3x_1 + 4x_2 = 0 \\ 3x_1 - 4x_2 = 0 \end{matrix}\right\} \; 3x_1 = 4x_2 \Rightarrow \frac{x_1}{4} = \frac{x_2}{3} = 1,$$ (sa

$$\Rightarrow \; x_1 = 4, x_2 = 3$$

$\therefore \; X = (4, 3)$ is the eigen vector corresponding to $\lambda = 6$.

Further to find eigen vector corresponding to $\lambda = -1$:

Put $\lambda = -1$ in (2)

$$\Rightarrow \begin{bmatrix} 3+1 & 4 \\ 3 & 2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} 4x_1 + 4x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{matrix}\right\} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2 \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = 1 \quad \text{(say)}$$

$$\Rightarrow \quad x_1 = -1, \; x_2 = 1$$

Thus $X = (-1, 1)$ is the eigen vector corresponding to $\lambda = -1$.

Hence $\lambda = 6$ & $X = (4, 3)$

$\lambda = -1$ & $X = (-1, 1)$ are the required eigen values & eigen vectors.

Example : 2

Find eigen values & eigen vectors of $A = \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$

Solution :

Consider $A - \lambda I = \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & 3 \\ 2 & 9-\lambda \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 3 \\ 2 & 9-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(9-\lambda) - 6 = 0$$

$$\Rightarrow 36 - 13\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 13\lambda + 30 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda - 3\lambda + 30 = 0$$

$$\Rightarrow (\lambda - 10)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 3, 10 \text{ are the eigen values.}$$

To find eigen vectors

Let $X = (x_1, x_2)$ be the required eigen vector

$\therefore \quad (A - \lambda I) X = 0$

$$\therefore \begin{bmatrix} 4-\lambda & 3 \\ 2 & 9-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

Case (i) For $\lambda = 3$, equation (1) gives

$$\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \left.\begin{matrix}x_1+3x_2=0\\2x_1+6x_2=0\end{matrix}\right\} x_1+3x_2=0 \Rightarrow x_1=-3x_2$$

$$\Rightarrow \frac{x_1}{+3} = \frac{x_2}{-1} = 1 \quad \text{(say)}$$

$$\therefore \; x_1 = +3,\; x_2 = -1$$

Thus $X = (3, -1)$ is the eigen vector corresponding to $\lambda = 3$.

Case (ii) For $\lambda = 10$, equation (1) gives

$$\begin{bmatrix}-6 & 3\\2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \left.\begin{matrix}-6x_1+3x_2=0\\2x_1-x_2=0\end{matrix}\right\} \Rightarrow 2x_1-x_2=0$$

$$\Rightarrow 2x_1 = x_2$$

$$\text{or} \quad \frac{x_1}{1} = \frac{x_2}{2} = 1 \text{ (say)}$$

$$\Rightarrow x_1 = 1,\; x_2 = 2$$

Thus $X = (1, 2)$ is the eigen vector corresponding to $\lambda = 10$.

Example : 3

Find the eigen values & eigen vectors of $\begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$

Solution :

Let

$$A = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$$

Consider $A - \lambda I = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2-\lambda & 0 & 1\\0 & 2-\lambda & 0\\1 & 0 & 2-\lambda\end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix}2-\lambda & 0 & 1\\0 & 2-\lambda & 0\\1 & 0 & 2-\lambda\end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda)(2-\lambda)^2 - 0 + 1\,[0-(2-\lambda)] = 0$

$\Rightarrow \quad (2-\lambda)\{(2-\lambda)^2 - 1\} = 0$

$\Rightarrow \quad (2-\lambda)(4+\lambda^2-4\lambda-1) = 0$

$\Rightarrow \quad (2-\lambda)(\lambda^2-4\lambda+3) = 0$

$\Rightarrow \quad 2-\lambda = 0,\ \lambda^2-4\lambda+3=0$

$\Rightarrow \quad \lambda = 2,\ (\lambda-3)(\lambda-1) = 0 \Rightarrow \lambda = 3, 1.$

Thus, $\lambda = 1, 2, 3$ are the eigen values.

To find eigen vectors

Let $X = (x_1, x_2, x_3)$ be the required eigen vector.

Consider

$$(A-\lambda I)X = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{--- (1)}$$

Case (i) For $\lambda = 1$, Equation (1) gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1+x_3 \\ x_2 \\ x_1+x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1+x_3=0 \\ x_2=0 \\ x_1+x_3=0 \end{matrix}$$

$$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3 \Rightarrow \frac{x_1}{1} = \frac{x_3}{-1} = 1 \text{ (say)}$$

$$\Rightarrow \quad x_1 = 1, x_3 = -1$$

Thus, $X = (x_1, x_2, x_3) \equiv (1, 0, -1)$ is the eigen vector corresponding to $\lambda = 1$.

Case (ii) For $\lambda = 2$, Equation (1) gives

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_3 \\ 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_3=0 \\ x_1=0 \end{matrix}$$

Since we donot have an equation for x_2, we choose the value of x_2 as 1.

Thus $X = (x_1, x_2, x_3) \equiv (0, 1, 0)$ is the eigen vector corresponding to $\lambda = 2$.

Case (iii) For $\lambda = 3$, Equation (1) gives

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} -x_1 + x_3 = 0 \\ -x_2 = 0 \\ x_1 - x_3 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x_2 = 0 \\ x_1 = x_3 \end{matrix} \Rightarrow \frac{x_1}{1} = \frac{x_3}{1} = 1$$ (sa

$$\Rightarrow x_1 = 1, x_3 =$$

Thus $X = (x_1, x_2, x_3) \equiv (1, 0, 1)$ is the eigen vector corresponding to $\lambda = 3$.

Example : 4

Find the eigen values & eigen vectors of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution :

$$A - \lambda I = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left\{(3-\lambda^2)-1\right\} + 2\left\{-2(3-\lambda)+2\right\} + 2\left\{2-2(3-\lambda)\right\} = 0$$

$$\Rightarrow (6-\lambda)(9+\lambda^2-6\lambda-1) + 2(-6+2\lambda+2) + 2(2-6+2\lambda) = 0$$

$$\Rightarrow (6-\lambda)(\lambda^2-6\lambda+8) + 2(2\lambda-4) + 2(2\lambda-4) = 0$$

$$\Rightarrow (6-\lambda)(\lambda-2)(\lambda-4) + 4(\lambda-2) + 4(\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)\left\{(6-\lambda)(\lambda-4)+8\right\} = 0$$

$$\Rightarrow (\lambda-2)\left\{6\lambda-24-\lambda^2+4\lambda+8\right\} = 0$$

$$\Rightarrow -(\lambda-2)\left(\lambda^2-10\lambda+16\right) = 0$$

$$\Rightarrow -(\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \lambda-2=0,\ \lambda-2=0,\ \lambda-8=0$$

$\Rightarrow$ $\lambda = 2, 2, 8$ are the required eigen values.

To Find eigen vectors: Let $X = (x_1, x_2, x_3)$ be such that

$$(A - \lambda I)\,X = 0 \Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) For $\lambda = 2$, Equation (1) becomes

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 4x_1 - 2x_2 + 2x_3 = 0 \\ -2x_1 + x_2 - x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{matrix}\right\} 2x_1 - x_2 + x_3 = 0$$

By inspection $x_1 = 1,\ x_2 = 0, x_3 = -2$

or $\quad x_1 = 1,\ x_2 = 2,\ x_3 = 0$

Thus $X = (1, 0, -2)$ & $(1, 2, 0)$ are the eigen vectors corresponding to $\lambda = 2$.

Case (ii) For $\lambda = 8$, Equation (1) gives

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} -2x_1 - 2x_2 + 2x_3 = 0 \\ -2x_1 - 5x_2 - x_3 = 0 \\ +2x_1 - x_2 - 5x_3 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1 + x_2 - x_3 = 0 \\ 2x_1 + 5x_2 + x_3 = 0 \\ 2x_1 - x_2 - 5x_3 = 0 \end{matrix}$$

By the rule of cross-multiplication

$$\frac{x_1}{\begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x}{6} = \frac{x_2}{-3} = \frac{x_3}{3} = k \text{ (say)}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k = 1 \text{ (say)}$$

$$\Rightarrow \quad x_1 = 2,\ x_2 = -1,\ x_3 = 1$$

Thus $X = (2, -1, 1)$ is the eigen vector corresponding to $\lambda = 8$

Example : 5

Find eigen values and eigen vectors of $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution :

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

The characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left\{(2-\lambda)^2-1\right\}+1\left\{-(2-\lambda)+1\right\}+1\left\{1-(2-\lambda)\right\}=0$$

$$\Rightarrow (2-\lambda)\left\{4+\lambda^2-4\lambda-1\right\}+(-2+\lambda+1)+(1-2+\lambda)=0$$

$$\Rightarrow (2-\lambda)\left(\lambda^2-4\lambda+3\right)+(\lambda-1)+(\lambda-1)=0$$

$$\Rightarrow (2-\lambda)(\lambda-1)(\lambda-3)+2(\lambda-1)=0$$

$$\Rightarrow (\lambda-1)\left\{(2-\lambda)(\lambda-3)+2\right\}=0$$

$$\Rightarrow (\lambda-1)\left\{2\lambda-6-\lambda^2+3\lambda+2\right\}=0$$

$$\Rightarrow -(\lambda-1)\left(\lambda^2-5\lambda+4\right)=0$$

$$\Rightarrow -(\lambda-1)(\lambda-4)(\lambda-1)=0$$

$$\Rightarrow \lambda-1=0, \lambda-4=0, \lambda-1=0$$

$$\Rightarrow \lambda=1, 4, 1$$

$\therefore$ The required eigen values are $\lambda = 1, 1, 4$

To find eigen vectors: Let $X=(x_1, x_2, x_3)$ be such that

$$(A-\lambda I)X = 0 \Rightarrow \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

se (i) For $\lambda = 1$, Equation (1) gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1-x_2+x_3=0 \\ -x_1+x_2-x_3=0 \\ x_1-x_2+x_3=0 \end{matrix}$$

By the rule of cross - multiplication

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}} = k \text{ (say)}$$

$$\Rightarrow \quad \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0} = k \Rightarrow x_1 = 0, x_2 = 0, x_3 = 0.$$

Thus $X = (0, 0, 0)$ is the eigen vector corresponding to $\lambda = 1$.

Case (ii) For $\lambda = 4$, Equation (1) gives

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{r} -2x_1 - x_2 + x_3 = 0 \\ -x_1 - 2x_2 - x_3 = 0 \\ x_1 - x_2 - 2x_3 = 0 \end{array}$$

By the rule of cross-multiplication.

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix}} = k \text{ (say)}$$

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3} = k$$

$$\text{or} \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = k_1 \qquad \text{(where } k_1 = 3k\text{)}$$

$$\Rightarrow \quad x_1 = k_1, x_2 = -k_1, x_3 = k_1$$

Thus, $X = (k_1, -k_1, k_1)$ is the eigen vector corresponding to $\lambda = 4$.

Note:

In the above eigen vector one can choose $k_1 = 1$ for simplicity so that $X = (1, -1, 1)$.

Example : 6

Show that the eigen values of A and A' are same

Solution :

We know that

$$(A - \lambda I)' = A' - \lambda I' \qquad (\because (A+B)' = A' + B')$$

$$= A' - \lambda I \qquad (\because I' = I)$$

$$\Rightarrow \quad |(A - \lambda I)'| = |A' - \lambda I|$$

$\Rightarrow |A-\lambda I| = |A'-\lambda I|$ $\quad (\because |A| = |A'|$

Thus $|A-\lambda I| = 0 = |A'-\lambda I|$

i.e., Characteristic equation of A and A' is one and same

$\Rightarrow$ Characteristic roots of A and A' are same.

Hence A and A' have same eigen values.

***Example* : 7**

Show that the matrices A and $B^{-1}AB$ have the same eigen values.

Solution :

Let $B^{-1}AB = C$

$\Rightarrow C - \lambda I = B^{-1}AB - \lambda I$

$= B^{-1}AB - B^{-1}(\lambda I)\,B$ $\quad (\because B^{-1}(\lambda I)B = B^{-1}(\lambda)B = \lambda B^{-1}B = \lambda$

$= B^{-1}(A - \lambda I)\,B.$

$\Rightarrow |C - \lambda I| = |B^{-1}(A-\lambda I)\,B|$

$= |B^{-1}| \cdot |A-\lambda I| \cdot |B|$

$= |A-\lambda I| \cdot |B^{-1}| \cdot |B|$

$= |A-\lambda I| \cdot |B^{-1}B|$

$= |A-\lambda I| \cdot |I|$

$= |A-\lambda I|$ $\quad (\because |I| = 1$

$\Rightarrow \quad |C-\lambda I| = 0 = |A-\lambda I|$

i.e., Characteristics equation of A and C is one and same.

Hence A and $C = B^{-1}AB$ have the same eigen values.

***Example* : 8**

Eigen value of a matrix is zero if and only if the matrix is singular.

Solution :

Let A be any matrix

First suppose that 0 is eigen value of A.

$\Rightarrow$ $\lambda = 0$ satifies the equation $|A-\lambda I| = 0$

$\Rightarrow \quad |A - 0 \cdot I| = 0$

$$\Rightarrow \quad |A-0| = 0$$

$$\Rightarrow \quad |A| = 0$$

$\Rightarrow$ A is singular.

Conversely suppose that,

A is singular $\Rightarrow |A| = 0$

$$\Rightarrow |A - 0\cdot I| = 0$$

$\Rightarrow \lambda = 0$ satisfies the equation $|A - \lambda I| = 0$

$\Rightarrow \lambda = 0$ is the eigen value.

Example : 9

Characteristic roots of a triangular matrix are just the diagonal elements.

Solution :

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ 0 & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

$$\Rightarrow A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ 0 & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 1 & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & a_{nn}-\lambda \end{bmatrix}$$

$$\Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

This on expansion gives.

$(a_{11}-\lambda)(a_{22}-\lambda)\,\ldots\ldots\,(a_{nn}-\lambda)=0.$

$\Rightarrow \qquad \lambda = a_{11}, a_{22}, \ldots\ldots, a_{nn}.$

Hence the eigen value of a triangular matrix are just the diagonal elements.

Exercise

Find the eigen values of the following

1. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 2. $\begin{bmatrix} a & m & n \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Find the eigen values & eigen vectors of the following

6. $\begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$ 7. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 8. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ 10. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ 11. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

12. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ 13. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Answers

1. $\lambda = 0, 3, 15.$
2. $\lambda = a, b, c.$
3. $\lambda = 3, -1.$
4. $\lambda = 1, 1, 1.$
5. $\lambda = 1, 1, 1.$

6. $\lambda = -1, 3$, $X = (1, -1)$ for $\lambda = -1$ & $X = (1, -5)$ for $\lambda = 3$.

7. $\lambda = 0, 3, 15$, $X = (1, 2, 2)$ for $\lambda = 0$; $X = (2, 1, -2)$ for $\lambda = 3$ & $X = (2, -2, 1)$ for $\lambda = 15$.

8. $\lambda = 5, -3, -3$, $X = (1, 2, -1)$ for $\lambda = 5$, $X = (2, -1, 0)$ for $\lambda = -3$.

9. $\lambda = 1, \sqrt{5}, -\sqrt{5}$, $X = (0, 0, 0)$ for $\lambda = 1$; $X = (2, 1-\sqrt{5}, 0)$ for $\lambda = \sqrt{5}$ & $X = (2, 1+\sqrt{5}, 0)$ for $\lambda = -\sqrt{5}$.

10. $\lambda = 2, 3, -1$; $X = (3, 1, 1)$ for $\lambda = 2$
$X = (-4, 1, -3)$ for $\lambda = 3$
$X = (0, 5, 5)$ for $\lambda = -1$

11. $\lambda = 1, -4, 7$; $X_1 = (k, 0, 0)$; $X_2 = (2k, -5k, 0)$; $X_3 = \left(\frac{27k}{60}, \frac{2k}{11}, k\right)$.

12. $X_1 = (2k, -k, k)$; $X_2 = (k_1, k_2, k_2 - 2k_1)$.

13. $X_1 = (k, 2k, 2k)$; $X_2 = \left(k, \frac{k}{2}, -k\right)$; $X_3 = \left(k, -k, \frac{k}{2}\right)$.

2.9 Cayley-Hamilton theorem

Statement : Every square matrix satisfies its characteristic equation.

or

If $|A - \lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots\dots\dots + a_{n-1} \lambda + a_n = 0$ is a characteristic equation of A, then

$$a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots\dots\dots + a_{n-1} A + a_n = 0$$

Proof

We know that,

$$AA^{-1} = I \Rightarrow A \cdot \frac{adj\,(A)}{|A|} = I$$

$$\Rightarrow A \cdot adj(A) = |A| \cdot I$$

$$\therefore \quad (A - \lambda I)\, adj\,(A - \lambda I) = |A - \lambda I| \cdot I \qquad \text{---- (1)}$$

Suppose $adj\,(A - \lambda I) = B$ so that

$$(1) \Rightarrow (A - \lambda I)\, B = |A - \lambda I| \cdot I$$

$$= \left(a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots\dots\dots + a_{n-1} \lambda + a_n\right) I$$

$$\Rightarrow \qquad (A - \lambda I)\left(b_0 \lambda^{n-1} + b_1 \lambda^{n-2} + \dots\dots\dots + b_{n-2} \lambda + B_{n-1}\right)$$

$$(a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots + a_{n-1}\lambda + a_n)I$$

(Since RHS is of degree n; B must be a polynomial of degree $n-1$)

Equating the coefficient of like terms we have

$Ab_{n-1} = a_n I \;\Rightarrow\; Ab_{n-1} = a_n I$

$Ab_{n-2} - b_{n-1} I = a_{n-1} I \;\Rightarrow\; Ab_{n-2} - b_{n-1} = a_{n-1} I$

$Ab_{n-3} - b_{n-2} I = a_{n-2} I \;\Rightarrow\; Ab_{n-3} - b_{n-2} = a_{n-2} I$

....

$Ab_0 - b_1 I = a_1 I \;\Rightarrow\; Ab_0 - b_1 = a_1 I$

$-b_0 I = a_0 I \;\Rightarrow\; -b_0 = a_0 I$

Premultiplying bothsides of the above equations respectively by $I, A, A^2, \ldots\ldots, A^{n-1}, A^n$ we get

$$Ab_{n-1} = a_n I$$
$$A^2 b_{n-2} - A b_{n-1} = A\, a_{n-1} I$$
$$A^3 b_{n-3} - A^2 b_{n-2} = A^2 a_{n-2} I$$

....

$$A^n b_0 - A^{n-1} b_1 = A^{n-1} a_1 I$$
$$-A^n b_0 = A^n a_0 I$$

Adding all of these equations we get

$$0 = a_n I + A a_{n-1} I + A^2 a_{n-2} I + \ldots\ldots + A^{n-1} a_1 I + A^n a_0 I$$

($\because$ all the terms on LHS cancel with each other)

i.e., $a_0 A^n + a_1 A^{n-1} + \ldots\ldots + a_{n-2} A^2 + a_{n-1} A + a_n = 0$ ---- (1)

Hence the square matrix A satisfies the characteristic equation.

Note : Cayley-Hamelton theorem can be used to find the **inverse** of a non-singular matrix as given below :

On premultiplying (1) by A^{-1} we get

$$a_0 A^{-1} A^n + a_1 A^{-1} A^{n-1} + \ldots\ldots + a_{n-1} A^{-1} A + a_n A^{-1} = 0.$$

$$\Rightarrow \quad a_0 A^{n-1} + a_1 A^{n-2} + \ldots\ldots + a_{n-1} I + a_n A^{-1} = 0.$$

$$\Rightarrow \quad A^{-1} = -\frac{1}{a^n}\left\{a_0 A^{n-1} + a_1 A^{n-2} + \ldots\ldots + a_{n-1} I\right\}.$$

WORKED EXAMPLES

xample : 1

Verify Cayley Hamelton theorem and hence find the inverse of $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

olution :

Let $$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$$\Rightarrow \quad |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)\left[(2-\lambda)^2 - 1\right] + 1\left[-2+\lambda+1\right] + 1\left[1-2+\lambda\right] = 0$$

$$\Rightarrow \quad (2-\lambda)\left(3 - 4\lambda + \lambda^2\right) - 1 + \lambda - 1 + \lambda = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ is the characteristic equation of A.

To show that A satisfies this equation, we need to prove $A^3 - 6A^2 + 9A - 4I = 0$.

Consider,

$$A^2 = AA = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 12+5+5 & -10-6-5 & 10+5+6 \\ -6-10-5 & 5+12+5 & -5-10-6 \\ 6+5+10 & -5-6-10 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -2 \\ 21 & -21 & 22 \end{bmatrix}$$

Consider $A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9+0 \\ -21+30-9+0 & 22-36+18-4 & -21+30-9+0 \\ 21+30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad A^3 - 6A^2 + 9A - 4I = 0$ ---- (1

Hence the square matrix A satifies its characteristic equation.

To find A^{-1} : Premultiplying (1) by A^{-1} we get

$A^{-1}A^3 - 6A^{-1}A^2 + 9A^{-1}A - 4A^{-1}I = 0$

$\Rightarrow \; A^2 - 6A + 9I - 4A^{-1} = 0$

$$\Rightarrow \quad A^{-1} = \frac{1}{4}\{A^2 - 6A + 9I\} = \frac{1}{4}\left\{\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{4}\begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3/4 & 1/4 & -1/4 \\ 1/4 & -3/4 & 1/4 \\ -1/4 & 1/4 & -3/4 \end{bmatrix}.$$

Example : 2

Verify Cayley-Hamilton theorem for $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{bmatrix}$$

$$\Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)(2-\lambda)(3-\lambda) + 2(-2)(2-\lambda) = 0$

$\Rightarrow \quad (1-\lambda)(6 - 5\lambda + \lambda^2) - 8 + 4\lambda = 0$

$\Rightarrow \quad 6 - 5\lambda + \lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 - 8 + 4\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 7\lambda - 2 = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$, this is a characteristic equation of A.

To show that $A^3 - 6A^2 + 7A + 2I = 0$, consider

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Consider,

$$A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0 & 34-48+14 \\ 12-12 & 8-24+14+2 & 23-30+7 \\ 34-48+14 & 0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e., $A^3 - 6A^2 + 7A + 2I = 0$

Thus, A satisfies the characteristic equation.

Hence the Cayley-Hamilton theorem is verified.

Example : 3

Show that $\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ satifies the Cayley Hamilton theorem and hence find its invers

Solution :

Let $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{bmatrix}$$

$$\Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (7-\lambda)\left[(7-\lambda)(4-\lambda) - 4\right] - 4\left[4(4-\lambda) - 4\right] - 1\left[-16 + 4(7-\lambda)\right] = 0.$$

$$\Rightarrow \quad (7-\lambda)\left[24 - 11\lambda + \lambda^2\right] - 48 + 16\lambda - 12 + 4\lambda = 0.$$

$$\Rightarrow \quad 168 - 77\lambda + 7\lambda^2 - 24 + 11\lambda^2 - \lambda^3 - 60 + 20\lambda = 0$$

$\Rightarrow \quad -\lambda^3 + 18\lambda^2 - 81\lambda + 108 = 0$

$\Rightarrow \quad \lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$, is the characteristic equation.

To show that $A^3 - 18A^2 + 81A - 108I = 0$, consider

$$A^2 = AA = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 783 & 756 & -189 \\ 756 & 783 & -189 \\ -756 & -756 & 216 \end{bmatrix}$$

Consider

$$A^3 - 18A^2 + 81A - 108\,I = \begin{bmatrix} 783 & 756 & -189 \\ 756 & 783 & -189 \\ -756 & -756 & 216 \end{bmatrix} - 18\begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$$+81\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} - 108\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 18A^2 + 81A - 108I = 0$. ---- (1)

Hence Cayley Hamilton theorem is verified.

To find A^{-1} : Permultiply (1) by A^{-1} to get.

$A^{-1}A^3 - 18A^{-1}A^2 + 81A^{-1}A - 108A^{-1}I = 0$

$\Rightarrow \quad A^2 - 18A + 81I = 108A^{-1}$

$\Rightarrow \quad A^{-1} = \frac{1}{108}\left\{A^2 - 18A + 81\,I\right\}$

$$= \frac{1}{108}\left\{\begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - 18\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 81\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{108}\begin{bmatrix} 24 & -12 & 3 \\ -12 & 24 & 3 \\ 12 & 12 & 33 \end{bmatrix}$$

This is the required inverse of A.

***Example* : 4**

Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and hence find A^2, A^3 and also establish

$2A^5 - 3A^4 + A^2 - 4I = 138A - 403I$

Solution :

Let $\quad A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}$$

$$\Rightarrow \quad |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda) + 1 = 0 \Rightarrow 6 - 5\lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda + 7 = 0$$

This is the characteristic equation of A.

Now to show that $A^2 - 5A + 7I = 0$, consider

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Consider $A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

i.e., $\quad A^2 - 5A + 7I = 0$ ---- (1)

Hence the Cayley-Hamilton theorem is verified.

Now (1) $\Rightarrow \quad A^2 = 5A - 7I$ ---- (2)

$\Rightarrow \quad A^3 = 5A^2 - 7AI = 5A^2 - 7A$ ---- (3) (Multiplied by A)

$\Rightarrow \quad A^4 = 5A^3 - 7A^2$ ---- (4) (Multiplied by A)

$\Rightarrow \quad A^5 = 5A^4 - 7A^3$ ---- (5) (Multiplied by A)

$\therefore (2) \Rightarrow A^2 = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$(3) \Rightarrow A^3 = 5\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 40 & 25 \\ -25 & 15 \end{bmatrix} - \begin{bmatrix} 21 & 7 \\ -7 & 14 \end{bmatrix} = \begin{bmatrix} 19 & 18 \\ -18 & 1 \end{bmatrix}$

Consider

$$\begin{aligned} LHS &= 2A^5 - 3A^4 + A^2 - 4I \\ &= 2\left(5A^4 - 7A^3\right) - 3A^4 + A^2 - 4I \quad \text{(by (5))} \\ &= 7A^4 - 14A^3 + A^2 - 4I \\ &= 7\left(5A^3 - 7A^2\right) - 14A^3 + A^2 - 4I \quad \text{(by (4))} \\ &= 21A^3 - 48A^2 - 4I \\ &= 21\left(5A^2 - 7A\right) - 48A^2 - 4I \\ &= 57\,A^2 - 147A - 4I \\ &= 57\,(5A - 7I) - 147A - 4I \\ &= 138A - 403I = RHS \end{aligned}$$

Hence $2A^5 - 3A^4 + A^2 - 4I = 138A - 403I$.

Example : 5

Verify Cayley-Hamilton theorem for $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{bmatrix}$$

$$\Rightarrow \quad |A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left(\lambda^2+a^2\right) - c\left(c\lambda - ab\right) - b\left(ac + b\lambda\right) = 0$$

$$\Rightarrow -\lambda^3 - \lambda a^2 - \lambda c^2 + abc - abc - \lambda b^2 = 0$$

$\Rightarrow \lambda^3 + \lambda\left(a^2+b^2+c^2\right) = 0$, is the characteristic equation.

Now to show that $A^3 + A\left(a^2+b^2+c^2\right) = 0$, consider

$$A^2 = A \cdot A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\left(b^2+c^2\right) & ab & ac \\ ab & -\left(a^2+c^2\right) & bc \\ ac & bc & -\left(a^2+b^2\right) \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} -\left(b^2+c^2\right) & ab & ac \\ ab & -\left(a^2+c^2\right) & bc \\ ac & bc & -\left(a^2+b^2\right) \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3 - b^2c - a^2c & bc^2 + b^3 + a^2b \\ a^2c + b^2c + c^3 & 0 & -ab^2 - ac^2 - a^3 \\ -a^2b - b^3 - bc^2 & ac^2 + ab^2 + a^3 & 0 \end{bmatrix}$$

$$= -\left(a^2+b^2+c^2\right)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

i.e., $\quad A^3 = -\left(a^2+b^2+c^2\right)A$

$\Rightarrow \quad A^3 + \left(a^2+b^2+c^2\right)A = 0$

Hence A satisfies the Cayley-Hamilton theorem.

Exercise

1. Verify the Cayley-Hamilton theorem for $\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$.

2. Verify the Cayley-Hamilton theorem for $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

3. Show that $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 3 & -1 & 1 \end{bmatrix}$ satisfies the Cayley-Hamilton theorem.

4. Show that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ satisfies the Cayley-Hamilton theorem.

5. Show that $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satisfies the Cayley-Hamilton theorem.

6. Verify the Cayley-Hamilton theorem for $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

7. Find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ using Cayley-Hamilton theorem.

8. Find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ using Cayley-Hamilton theorem.

Answers

1. $\frac{1}{4}\begin{bmatrix} 5 & 0 & 0 \\ -3 & 4 & 0 \\ 2 & -1 & 1 \end{bmatrix}$.

8. $\frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$.