CHAPTER 5
CONNECTIVITY

5.1 Introduction

The connectivity of a graph is a particularly intuitive area of Graph Theory. The connectivity extends the concepts of cutvertex, bridge and block. The vertex connectivity and edge connectivity are useful in deciding which of two graphs is more connected.

5.2 Vertex Connectivity and Edge Connectivity

Let G be a connected graph. We have known that if G contains a vertex v such that $G-v$ is disconnected, then v is a cutvertex of G. Also if G contains an edge e such that $G-e$ is disconnected, then e is a bridge of G. We now consider extensions of these two concepts.

Definition: A subset U of the vertex set of a connected graph G is said to be a **vertex cutset** of G if $G-U$ is disconnected provided removal of no proper subset of U disconnects G.

Definition : A subset W of the edge set of a connected graph G is said to be an **edge cutset** of G if $G-W$ is disconnected provided removal of no proper subset of W disconnects G.

Example 1. Find (i) two different vertex cutsets and (ii) two different edge cutsets of the following graph G.

G:

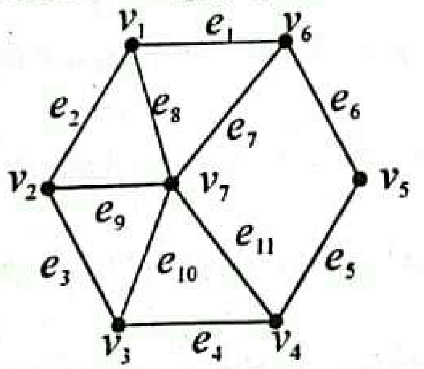


Figure 5.1

Solution: Let $U_1 = \{v_5, v_7, v_3\}$ and $U_2 = \{v_4, v_6\}$. Then $G-U_1$ and $G-U_2$ are both disconnected and removal of no proper subsets of U_1 and U_2 disconnect G. Therefore U_1 and U_2 are vertex cutsets.

Let $W_1 = \{e_1, e_8, e_9, e_3\}$ and $W_2 = \{e_5, e_6\}$. Then $G-W_1$ and $G-W_2$ are both disconnected; and W_1 and W_2 do not contain proper edge cutsets. Hence W_1 and W_2 are both edge cutsets.

We note that

1. The set of vertices $\{v_3, v_4, v_6\}$ is not a vertex cutset of G, because one of its proper subsets $\{v_4, v_6\}$ is a vertex cutset.
2. The set of edges $\{e_4, e_5, e_6\}$ is not an edge cutset of G, since one of its proper subsets $\{e_5, e_6\}$ is an edge cutset.
3. There are many other vertex cutsets and edge cutsets in G.

We note that an edge cutset always cuts a graph into two components.

Example 2. Show that every edge of a tree is an edge cutset.

Solution: Let T be a nontrivial tree. Then the removal of any edge e from T breaks the tree T into two parts. Thus $\{e\}$ is an edge cutset of T.

Example 3. Show that if G is a connected graph and U is an edge cutest with $\lambda(G)$ edges, then $G-U$ contains exactly two components.

Solution: Let G be a connected graph. Let U be an edge cutset such that $\lambda(G)=|U|$. Thus removal of $\lambda(G)$ edges from G results in a disconnected graph. Since an edge cutset always cuts a graph into two components, it follows that $G-U$ contains exactly two components.

Example 4. Prove or disprove : If G is a connected graph and S is a vertex cutest with $\kappa(G)$ vertices, then $G-S$ contains exactly two components.

Solution: This statement is not true always. For example, consider the graph $G=K_{1,4}$. Then since $K_{1,4}$ has a cut vertex, say v, it follows that $\kappa(K_{1,4}) = 1$ and $S = \{v\}$. Clearly $\kappa(K_{1,4}) = |S|$. The removal of S vertices from G produces a disconnected graph with 4 components. Thus $G-S$ contains more than two components.

Example 5. Show that every bridge of a graph is an edge cutset.
Solution: Let G be a connected graph and e be any bridge of G. The deletion of e from G results in a disconnected graph. Thus $\{e\}$ is a cutset of G.

Definition: An edge cutset with respect to a pair of vertices a and b in a connected graph G **separates** vertices a and b such that a and b lie into two different components.

Example 6. Find all edge cutsets with respect to the vertex pair v_2, v_3 in the following graph.

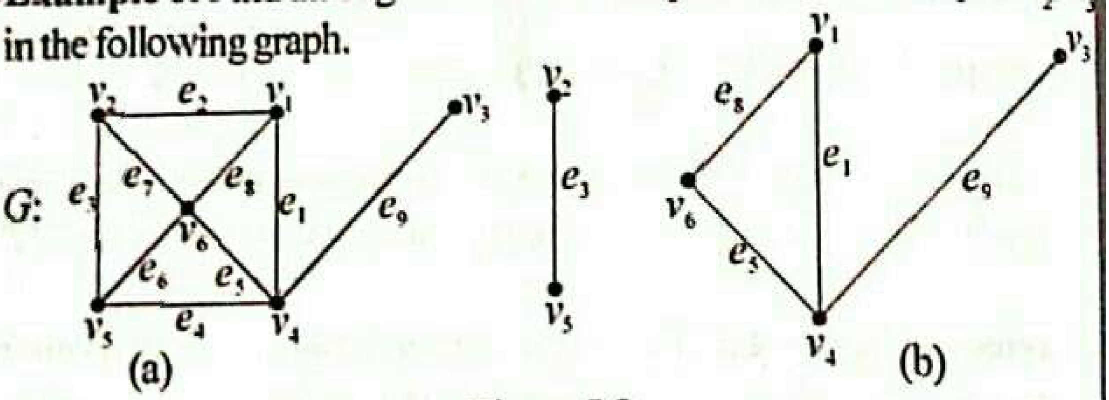


Figure 5.2

Solution: The set $\{e_2, e_7, e_6, e_4\}$ is an edge cutset with respect to the vertex pair v_2 and v_3. The removal of an edge cutest $\{e_2, e_7, e_6, e_4\}$ from a graph G results in a disconnected graph into two components, see Figure 5.2(b). The vertices v_2 and v_3 lie in two different components.

Similarly, we find other edge cutsets, They are

$\{e_2, e_3, e_7\}$, $\{e_2, e_8, e_5, e_4\}$, $\{e_3, e_7, e_8, e_1\}$, $\{e_3, e_6, e_5, e_1\}$ $\{e_1, e_5, e_4\}$, $\{e_9\}$.

These sets are edge cutsets with respect to the vertex pair v_2, v_3 in the graph G of Figure 5.2(a).

Example 7. Find all edge cutsets with respect to the vertex pair v_3, v_6 in the following graph G.

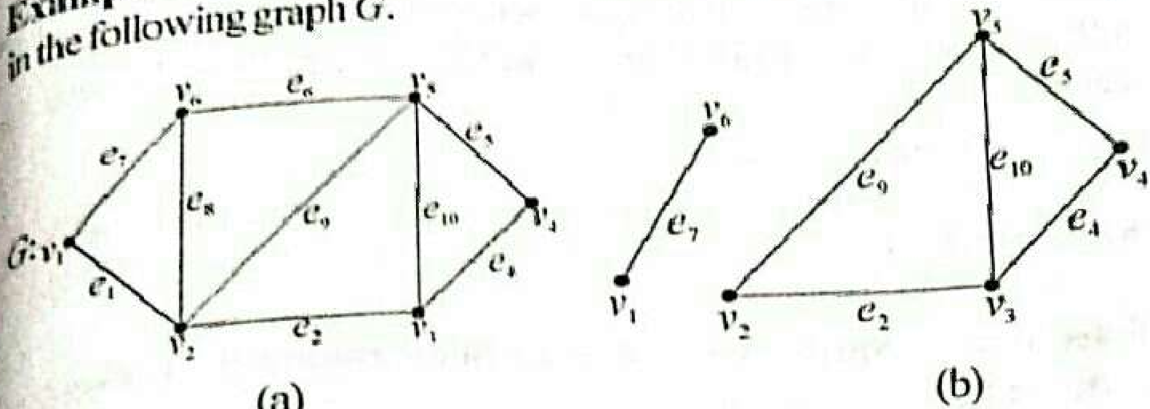


Figure 5.3

Solution: The set $\{e_1, e_8, e_6\}$ is an edge cutset with respect to the vertex pair v_3, v_6. The deletion of an edge cutset $\{e_1, e_8, e_6\}$ from a graph G produces a disconnected graph into two components, see Figure 5.3(b). The vertices v_3 and v_6 are in two different components.

The following sets are edge cutsets with respect to the vertex pair v_3, v_6 in the graph G of Figure 5.3 (a):

1. $\{e_2, e_6, e_9\}$
2. $\{e_2, e_5, e_{10}\}$
3. $\{e_2, e_4, e_{10}\}$.

Vertex Connectivity

Definition: The **vertex connectivity** or simply **connectivity** $\kappa(G)$ of a graph G is the minimum number of vertices who removal from G results in a disconnected or trivial graph.

Example 8. Show that a graph G has $\kappa(G) = 0$ if and only if G is a disconnected or trivial graph.

Solution: Suppose $\kappa(G) = 0$. Then G is disconnected or trivial.

Suppose G is a disconnected or trivial graph. Then clearly $\kappa(G) = 0$.

Example 9. Show that if C_p is a cycle with $p \geq 3$, then $\kappa(C_p) = 2$.

Solution: Suppose C_p is a cycle with $p \geq 3$. Then the deletion of any vertex from C_p results in a path P with p-1 vertices. We consider two cases.

Case 1. If $p=3$, then $P=K_2$. The removal of any vertex of K_2 results in a trivial graph.

Case 2. If $p \geq 4$, then P has a cut vertex. The removal any cut vertex of P results in a disconnected graph.

From the above two cases, we have $\kappa(C_p)=2$.

Example 10. Find the connectivity of a complete graph K_p.

Solution: The complete graph K_p with $p \geq 2$ vertices can not be disconnected by removing any number of vertices, but the deletion of $p-1$ vertices results in the trivial graph. Therefore

$$\kappa(K_p) = p-1 \text{ for } p \geq 2. \quad (1)$$

Consider the complete graph K_1. The removing the empty set of vertices from K_1 produces the trival graph. Therefore

$$\kappa(K_1)=0.$$

Thus from (1) and (2), we get

$$\kappa(K_p)=p-1 \text{ for } p \geq 1.$$

Example 11. Find the connectivity of $K_{m,n}$ where $1 \leq m \leq n$.

Solution: The vertex set V of $K_{m,n}$, $1 \leq m \leq n$, can be partitioned into two subsets V_1 and V_2 such that $|V_1|=m$ and $|V_2|=n$. Then the removal of $r < m$ vertices of V_1 from $K_{m,n}$ produces a connected graph. But the removal of m vertices of V_1 from $K_{m,n}$ produces the disconnected graph $\overline{K}_n$. Hence $\kappa(K_{m,n}) = m$.

Example 12. Show that the vertex connectivity of a nontrivial tree is one.

Solution: Let T be a nontrivial tree with p vertices. We consider the following two cases.

Case 1. Suppose $p=2$. Then $T = K_2$. The deletion of any vertex from K_2 produces the trivial graph. Thus $\kappa(T) = 1$.

Case 2. Suppose $p \geq 3$. Then T has a cut vertex v. The deletion of v from T produces a disconnected graph. Hence $\kappa(T)=1$.

From Cases 1 and 2, the result follows.

Edge Connectivity

Definition : The **edge connectivity** $\lambda(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected or trivial graph.

Example 13. Show that a graph G has $\lambda(G) = 0$ if and only if G is disconnected or trivial.

Solution : Suppose $\lambda(G) = 0$. Then G is a disconnected or trivial graph.

Conversely suppose G is disconnected. Then $\lambda(G)=0$. Suppose G is trivial. Then $G=K_1$. Clearly removing the empty set of edges from K_1 produces the trivial graph. Therefore $\lambda(K_1)= 0$.

Example 14. Show that if C_p is a cycle with $p \geq 3$, then $\lambda(C_p) = 2$.

Solution: Suppose C_p is a cycle with $p \geq 3$. Then the removal of any edge from C_p results in a path P with p vertices. Since every edge of P is a bridge, it follows that the removal of any edge from P results in a disconnected graph. Hence $\lambda(C_p) = 2$.

Example 15. Determine $\lambda(K_{m,n})$, where $i\leq m\leq n$.

Solution: The vertex set V of $K_{m,n}$ $(i\leq m\leq n)$ can be partitioned in two disjoint sets such that $|V_1| = m$ and $|V_2| = n$. Then the removal of $r < m$ edges from $K_{m,n}$ produces a connected graph. But removal of edges which are incident with any vertex of V_2 results in a disconnected graph. Then $\lambda(K_{m,n})=m$.

Example 16. Find the value of $\lambda(K_p)$.

Solution: Consider K_1.The addition of empty set of edges from K results in a trivial graph. Thus

$$\lambda(K_1)=0. \qquad (1)$$

Now consider K_p, $p\geq 2$. The removal of any $p-2$ edges of K produces a connected graph; but removal of $p-1$ edges which ar incident with any vertex of K_p produces a disconnected graph. Thus

$$\lambda(K_p)=p-1, \ p\geq 2. \qquad (2)$$

Thus from (1) and (2), we get

$$\lambda(K_p)= p-1, p\geq 1.$$

Example 17. Let G be a graph with $\varkappa(G) = 1$. What are the possibl values for the following numbers

(i) $\varkappa(G-v)$ *ii)* $\varkappa(G-e)$

iii) $\lambda(G-v)$ *iv)* $\lambda(G-e)$?

Solution: Let G be a graph with $\varkappa(G) = 1$.

Then G has cut vertices or $G=K_2$.

1) We now find the values of $\varkappa(G-v)$ and $\lambda(G-v)$.

If v is a cut vertex of G, then $G-v$ is disconnected.

Also if $G=K_2$, then $G-v$ is a trivial graph.

Hence $\varkappa(G-v)=0$

and $\lambda(G-v)=0$.

2) We now find the values of $\kappa(G-e)$ and $\lambda(G-e)$.

Suppose $G=K_2$. Then $G-e=\overline{K}_2$ which is disconnected.

In this case

$\kappa(G-e)=0$

and $\lambda(G-e)=0$.

Suppose $G\neq K_2$. Since $\kappa(G)=1$, G has a cutvertex. We now consider the following cases.

Case 1. If G has a bridge e, then $G-e$ is disconnected.

Thus $\kappa(G-e)=0$

$\lambda(G-e)=0$.

Case 2. If G has no bridge, then every edge e is on cycle. Therefore $G-e$ is connected and has a cut vertex.

Thus

$\kappa(G-e)=1$.

We see that $G-e$ may or may not contain a bridge.

Therefore

$\lambda(G-e)\geq 1$.

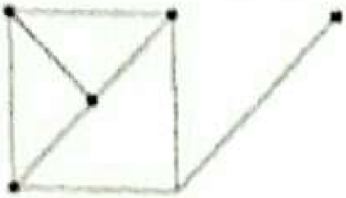
Exercise

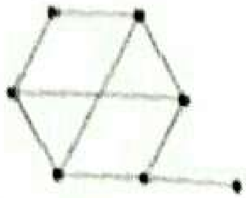
1. Show that a graph G is not complete, then $\kappa(G)$ is the minimum cardinality of a vertex cutset of G.
2. Show that a graph G is not trivial, then $\lambda(G)$ is the minimum cardinality of an edge cutset of G.
3. Find all edge cutsets with respect to the vertex pair v_5, v_7 in the graph G in Figure 5.1.
4. Find all edge cutsets with repect to the vertex pair v_1 and v_3 in the graph G in Figure 5.2(a).

5. Find the edge cutsets with respect to the vertex pair v_1, v_4 in the graph G in Figure 5.3(a).

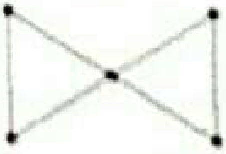
6. Show that if v is a vertex of a tree such that deg $v \geq 2$, then $\{v\}$ is a vertex cutset.

7. Find (i) two different vertex cutsets
and (ii) two different edge cutsets
each of the following graphs.

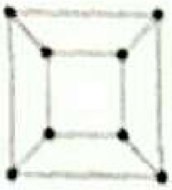
a) 

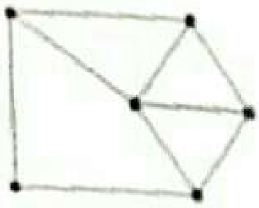
b) 

8. Find (i) the connectivity and (ii) the edge connectivity of each of the following graphs.

a) 

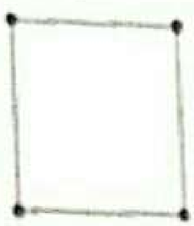
b) 

c) 

d) 

9. Show that (i) $\kappa(G) = \lambda(G)$
(ii) $\kappa(H) = \lambda(H)$
where graphs G and H are shown below:

G: 

H: 

10. Find the values of

i) $\kappa(K_6)$ ii) $\kappa(C_n)$

iii) $\kappa(K_9)$ iv) $\kappa(P_9)$

Determine the following

i) $\lambda(K_{5,6})$ ii) $\lambda(C_8)$

iii) $\lambda(K_7)$ iv) $\lambda(P_6)$.

Show that if G is a connected graph with a cutvertex, then $\kappa(G)=1$.

Show that if P is a nontrivial path, then $\kappa(P)=1$.

Show that if G is a connected graph with a bridge, then $\lambda(G)=1$.

Show that if P is a nontrivial path, then $\lambda(G)=1$.

Prove that if G is a connected cubic graph with a cutvertex, then $\kappa(G)=\lambda(G)$.

Prove that if P is a path, then $\kappa(G)=\lambda(G)$.

Prove that if C_p is a cycle with $p\geq 3$ vertices, then $\kappa(G)=\lambda(G)$.

Give an example of a graph with $\kappa(G)=\lambda(G)=3$.

Show that if G is a cubic graph, then $\kappa(G)=\lambda(G)$.

Show that the edge connectivity of a nontrivial tree is one.

Show tht a graph G has connectivity 1 if and only if $G=K_2$ or G is a connected graph with cut vertices.

Show that if G is a (p,q) graph and $q<p-1$, then $\kappa(G)=0$.