CHAPTER 2
SUBGRAPHS, PATHS AND CYCLES

2.1 Introduction

Some basic concepts about graph theory are discussed in Chapter 1. Here, we develop the theory a little further. In this chapter, we give a few more basic concepts of graph theory, such as subgraph, spanning subgraph, walk, path, cycle and bipartite graph. We also present some important results and several examples on these concepts.

2.2 Subgraphs, Spanning and Induced Subgraphs

Subgraphs

Definition: A **subgraph** H of a graph G is a graph having all of its vertices and edges of G.

Definition: If H is a subgraph of G, then G is a **supergraph** of H.

We note that a graph G is itself a subgraph of G.

Example 1. Draw two subgraphs of the following graph.

G:

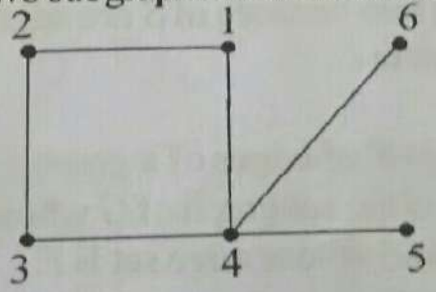
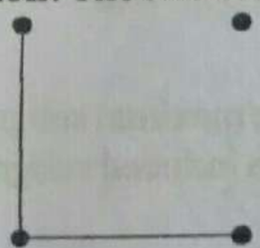


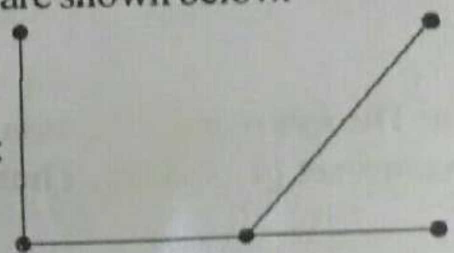
Figure 2.1

Solution: The two subgraphs of G are shown below.

G_1:



G_2:



Spanning Subgraphs

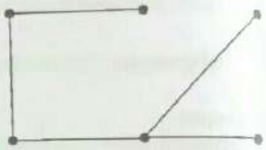
Definition: A **spanning subgraph** of a graph G is a subgraph containin all vertices of G.

Example 2. Draw two spanning subgraphs of the graph G shown Figure 2.1.

Solution: The following are two spanning subgraphs of G since eac of which contains all vertices of G.

G_3:

G_4:



Induced Subgraphs

Definition: The **induced subgraph** $<S>$ of a graph G is the maxima subgraph of G with vertex set S where S is any set of vertices of G.

It is clear that two vertices of S are adjacent in $<S>$ if an only if they are adjacent in G.

Definition: For any set F of edges of a graph G, the **edge induced subgraph** $<F>$ of G is the subgraph of G whose vertex set is the set of ends of edges of F and whose edge set is F.

Example 3. Draw an induced subgraph of the graph G shown in Figure 2.1.

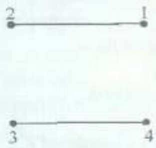
Solution: The following subgraph G_5 of G is the maximal subgraph of G with vertex set $\{1, 3, 4, 5\}$. Therefore G_5 is an induced subgraph of G.

G_5:

Example 4: Draw an edge induced subgraph of the graph G shown in Figure 2.1.

Solution: An edge induced subgraph G_6 of G is given below.

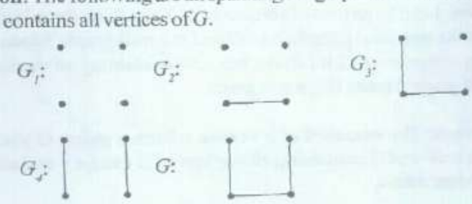
G_6:



Example 5. Find all spanning subgraphs of the following graph G.

G:

Solution: The following are all spanning subgraphs of G, since each of which contains all vertices of G.



Example 6. Show that $\bar{K}_p$ is a spanning subgraph of K_p.

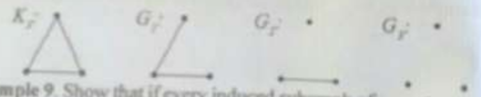
Solution. The graph $\bar{K}_p$ is a null graph with p vertices. It contains all vertices of K_p. Therefore $\bar{K}_p$ is a spanning subgraph of K_p.

Example 7. Show that an induced subgraph of a complete graph is complete.

Solution: Let V be a vertex set of a complete graph G. Let S be a subset of V. Then $<S>$ is an induced subgraph of G. Since every pair of vertices of G are adjacent, it follows that every pair of vertices of $<S>$ are adjacent. Hence $<S>$ is complete.

Example 8. Find all spanning subgraphs of K_3

Solution: The following are spanning subgraphs of K_3.



Example 9. Show that if every induced subgraph of a graph G with $p\geq 2$ vertices is a null graph, then G is a null graph.

Solution: Let S be any set of vertices of G. Then the induced subgraph $<S>$ is the maximal subgraph of G and is a null graph. Since S is an arbitary vertex set of G, it follows that $<S>$ containing all vertices of G is a null graph. Hence G is a null graph.

Definition: The **removal of a vertex** v from a graph G yields the subgraph $G-v$ of G containing all vertices of G except v and all edges not incident with v.

Definition: The **removal of an edge** e from a graph G yields the subgraph $G-e$ containing all vertices of G and all edges of G except e.

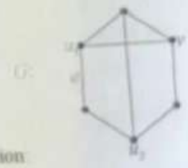
Definition: If u and v are not adjacent vertices in G, then the **addition of an edge** $e=uv$ yields the graph $G+e$ with the vertex set V and the edge set $E\cup\{e\}$.

We note that

1) $G-e$ is a spanning subgraph of G.
2) $G+e$ is the smallest supergraph of G.

Example 10. Draw the following subgraphs for the graph G shown below:

1) $G-v$
2) $G-e$ and also
3) Draw the graph $G+e_1$ where u_1 and u_2 are not adjacent vertices of G and $e_1=u_1u_2$.



Solution

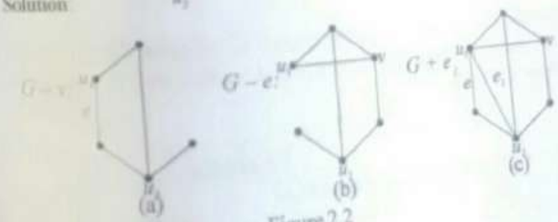


Figure 2.2

1) The removal of a vertex v from G produces the subgraph G… of G shown in Figure 2.2(a).
2) The removal of an edge e from G produces the subgraph G… of G shown in Figure 2.2(b).
3) The addition of an edge e_1 to the graph G gives the gra… $G+e_1$ shown in Figure 2.2(c). This graph is the smalle… supergraph of G.

Example 11. Show that the removal of a vertex v from $K_p (p\geq$… produces a complete graph K_{p-1}.

Solution: Let v be a vertex of $K_p (p\geq 2)$. Then $K_p - v$ is the subgraph … K_p and it is obtained by the removal of the vertex v from K_p togethe… with all edges incident with v. This implies that $K_p - v$ is a comple… graph with $p-1$ vertices. Thus

$$K_p - v = K_{p-1}$$

From Example 11, we see that if $G=K_p$, then G and $G-v$ a… both regular graphs. But this result is not true for any regular grap… Now we have the result.

Example 12. Show that, by an example, if G is regular, then $G-v$ … not regular.

Solution: Suppose G is a cycle. Then G is 2-regular. The removal o… any vertex v from G produces a path. Clearly a path is not regula… Thus $G-v$ is not regular.

Exercise

1. Draw all spanning subgraphs of
   i) K_4 ii) $K_4 - e$ iii) $K_4 - v$

2. Draw all spanning subgraphs of (C_p defined in section 2.3)
   i) C_4 ii) $C_5 - e$ iii) $C_5 - v$

3. Draw all spanning subgraphs of (P_p defined in section 2.3)
   i) P_3 ii) P_5

4. Draw all spanning subgraphs of ($K_{m,n}$ defined in section 2.5)
   i) $K_{1,2}$ ii) $K_{2,2} - e$ iii) $K_{1,3}$

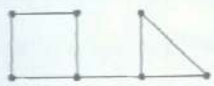
5. Show that every graph with p vertices is isomorphic to a subgraph of K_p.

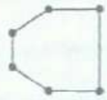
6. Show that if every induced subgraph of a graph G is complete, then G is complete.

7. Prove or disprove: If G is regular, then $G-v$ is regular.

8. Draw three different connected subgraphs of K_4.

9. Draw
   1) two spanning subgraphs
   2) two induced subgraphs of the following graphs.

i) 

ii) 

10. Define
    i) a spanning subgraph of a graph
    ii) an induced subgraph of a graph.

2.3 Paths and Cycles

Definition: Let u and v be vertices of graph G. A ***u-v walk*** of a gr[...] G is an alternating sequence of vertices and edges of G beginning w[...] vertex u and ending with vertex v, such that every edge is incident w[...] two vertices immediately proceeding and following it.

Definition: The vertices u and v are called the **initial** and **termin[...]** vertices of a u-v walk respectively and other vertices its internal vertic[...]

Definition: The number of occurences of edges in a walk is called [...] **length**.

Definition: A walk is called **trivial** if it has not edges.

Definition: A u-v walk is **closed** if $u=v$ and is open otherwise.

Definition: A walk is a **trail** if all the edges in it are distinct.

Definition: A walk is a **path** if all its vertices (and then necessarily [...] the edges) are distinct.

Clearly every path is a trail. But every trail need not be a pa[...]

Definition: A u-v path is **closed** if $u=v$.

Definition: A nontrivial closed path is a cycle.

A nontrivial closed path of a graph G is called a **cycle of G**

Definition: A cycle is odd or **even if** its length is odd or even.

Definition: A cycle of length n is an **n-cycle**.

Definition: A 3-cycle is called a **triangle**.

A graph of order p which is a path or cycle is denoted by P_p or C_p respectively.

Example 13. Give an example of a walk, a trail, a path and a cycle.

Solution:

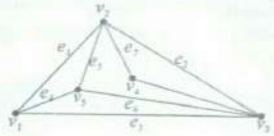


Figure 2.3

In the graph of Figure 2.3,

1) $v_1\ e_1\ v_2\ e_2\ v_3\ e_2\ v_2\ e_5\ v_5\ e_6\ v_3\ e_8\ v_4$ is a v_1 - v_4 walk which is not a trail, since e_2 appears twice.

2) $v_1\ e_1\ v_2\ e_5\ v_5\ e_4\ v_1\ e_3\ v_3\ e_8\ v_4$ is a v_1 - v_4 trail which is not a path, since v_1 appears twice.

3) $v_1\ e_1\ v_2\ e_2\ v_3\ e_6\ v_5$ is a v_1 - v_5 path.

4) $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_1$ is a v_1 - v_1 cycle.

We now present a result that relates walks and paths.

Theorem 2.1. Every u-v walk contains a u-v path.

Proof: Let W be a u-v walk in a graph G. If W is closed, the result is trivial. Thus suppose W: $u=u_0, u_1, u_2, \ldots, u_n=v$ is an open u-v walk of a graph G. It may be the case here that a vertex has received more than one label. If no vertex of G occurs in W more than once then W is ... v path. Thus, we assume that there are vertices of G which occur ... twice or more. Let j be the smallest positive integer such that th... exists $i < j$ with $u_i = u_j$. If the vertices $u_i, u_{i+1}, \dots, u_{j-1}$ are deleted fr... W, then a u-v walk W_1 is obtained having fewer vertices than that o... If no vertex is repeated in W_1, then W_1 is a u-v path. If this is not ... case, we continue the above procedure untils finally arriving at a ... walk which is a u-v path.

Theorem 2.2. If G is a graph with $\delta(G) \geq k$, then G has a pat... length k.

Proof: Let v_1 be a vertex of G. Choose v_2 adjacent to v_1. Since δ ... $\geq k$, it follows that there exist at least $k-1$ vertices other than v_1 wh... are adjacent to v_2. Choose $v_3 \neq v_1$ such that v_3 is adjacent to v_2... general, having chosen vertices $v_1, v_2, \dots, v_i$ where $1 < i \leq \delta(G)$, th... exists a vertex $v_{i+1} \neq v_1, v_2, \dots, v_i$ such that v_{i+1} is adjacent to v_i. T... process yields a path of length k in G.

Theorem 2.3. A closed walk of odd length contains a cycle.

Proof: Let $v = v_0, v_1, v_2, \dots, v_n = v$ be a closed walk of odd leng... Thus $n \geq 3$.

We prove the result by induction.

Suppose $n=3$. Then this walk is itself the cycle C_3. Hence t... result is true.

Assume the result is true for all walks of length less than n.

If the given walk of length n is itself a cycle, then the resul... true for n.

If not, there exist two positive integers i and j such that $i <$ ... $i \neq 0$, $j \neq n$, and $v_i = v_j$. Now

$$v_i, v_{i+1}, \dots, v_j$$

and

$$u = v_0, v_1, \dots, v_i, v_{j+1}, \dots, v_n = u$$

are closed walks contained in the given walk and sum of their lengths is n. Since n is odd, it follows that at least one of these walks is of odd length, which, by induction, contains a cycle.

Exercise

1. Give an example of a path of length 3 in K_4.
2. Give an example of a closed walk of even length which does not contain a cycle.
3. In a connected graph, prove that any two longest paths have a vertex in common.
4. Prove that a connected graph with $p \geq 2$ vertices is a nontrivial path if and only if

$$\sum_{i=1}^{p} \deg v_i = 4p - 6.$$

5. Show that if G is a graph with minimum degree $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G)+1$.
6. Define

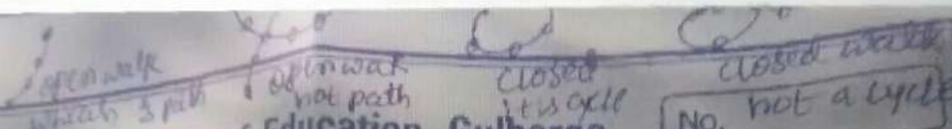
   i) a walk ii) a trail

   iii) a path iv) a cycle

   with an example.

2.4 Connectedness and Component

Definition: Two vertices u and v of a graph G are said to be **connected** if there exists a u-v path in G.



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Circuit → A closed walk with atleast one edge in which no vertex except the terminal vertices appears more than once is called a cycle.

Ex → $v_1 e v_2 e_8 v_7 e_9 v_1$ & $v_2 e_4 v_6 e_5 v_4 e_3 v_3 e_2 v_2$

Note → ~~A ve~~ In walks, path & circuit no edge appears more then once

2) A vertex can appear more than once in a walk but not in a path

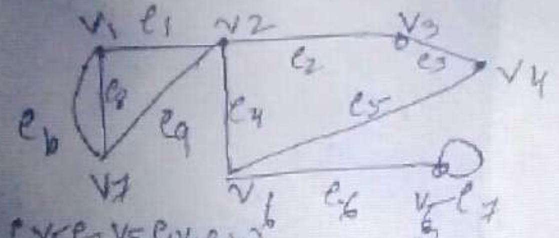
3) a path is an open walk, but an open walk need not be a path

4) A cycle is closed walk but a closed walk need not be a cycle

Walk → walk is defined as a finite alternating sq of vertices and edges of the form $v_i e_j$ which begins & ends with vertices

no edge appears more then once in the sq. Such a sq is called a walk or a trail in G.

Ex → $v_2 e_4 v_6 e_5 v_4 e_3 v_3$ & $v_1 e_8 v_2 v_6 e_6 v_5 e_7 v_5$



($v_5 e_7 v_5 e_6 v_6 e_5 v_4$)

2) Open walk & closed walk ($v_1 e_9 v_7 e_8 v_2 e_1 v_1$)

A walk that is not closed → open walk.

path → In a walk a vertex can appear more then once. An open walk in which no vertex appears more then once is called a simple path or a path

Ex → $v_6 e_5 v_4 e_3 v_3 e_2 v_2$ is a path whereas $v_5 e_7 v_5 e_6 v_6$ is an open walk but not a path

2.5 Bipartite Graphs

Definition: A graph G is called a **bipartite graph** or **bigraph** if the vetex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex of V_1 to a vertex of V_2.

Definition: A **complete bipartite graph** is a bipartite graph with vertex partition V_1 and V_2 such that each vertex of V_1 is joined to every vertex of V_2. If V_1 contains m vertices and V_2 contains n vertices, then the complete bipartite graph is denoted by $K_{m,n}$. The graph $K_{1,n}$ is a **star.**

Example 22. Show that the graph G of Figure 2.5(a) is a bipartite graph.

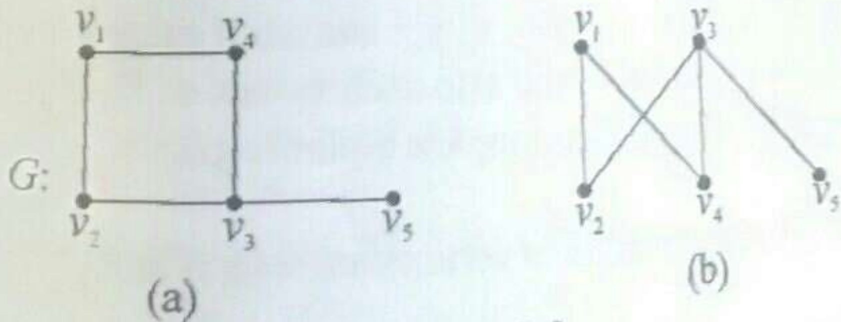


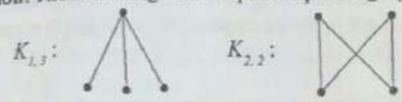
Figure 2.5

Solution: The vertex set V of G can be partitioned into two subsets $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4, v_5\}$ and each edge of G joins a vertex of

V_1 to a vertex of V_2. This graph is redrawn in Figure 2.5 (b) with V_1 at the top and V_2 at the bottom. This shows that G is bipartite.

Example 23. Draw complete bipartite graphs $K_{1,3}$, $K_{2,2}$.

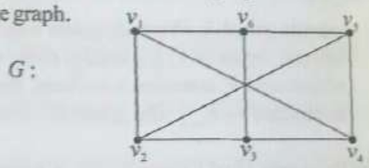
Solution: The following are complete bipartite graphs $K_{1,3}$ and $K_{2,2}$



Example 24. Draw all complete bipartite graphs with 5 vertices.

Solution: The following are complete bipartite graphs with 5 vertices.

Example 25. Show that the graph G shown below is a complete bipartite graph.



Solution: The vertex set V of G can be partitioned into two disjoint sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and each edge in G joins a vertex of V_1 to a vertex of V_2 and also each vertex of V_1 is joined to every vertex of V_2. Thus G is a complete bipartite graph.

Example 26. Find the number of vertices and edges in $K_{m,n}$.

Solution: The graph $K_{m,n}$ is a complete bipartite graph with vertex partition V_1 and V_2. Then the number of vertices in V_1 is m and that in V_2 is n. Thus $K_{m,n}$ has $m+n$ vertices.

The graph $K_{m,n}$ has vertex set $V=\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}\}$ which can be partitioned into two disjoint subsets $V_1 = \{v_1, \ldots, v_m\}$ and $V_2=\{v_{m+1}, \ldots, v_{m+n}\}$. Then $\deg v_i = n$ for $1 \le i \le m$ and $\deg v_i = m$ for $m+1 \le i \le m+n$.

$$\sum_{i=1}^{m+n} \deg v_i = \sum_{i=1}^{m} \deg v_i + \sum_{i=m+1}^{m+n} \deg v_i$$
$$= mn + nm$$
$$= 2mn$$
$= $ twice the number of edges (by Theorem 1.1.).

Therefore $K_{m,n}$ has mn edges.

Example 27. How many vertices and how many edges are there in complete bipartite graphs $K_{4,7}$ and $K_{7,11}$?

Solution: It is known that if $K_{m,n}$ is complete bipartite graph, then it has $m+n$ vertices and mn edges. Therefore $K_{4,7}$ has 11 vertices and 28 edges. Also the complete bipartite graph $K_{7,11}$ has 18 vertices and 77 edges.

Example 28. Let G be a graph with p vertices and q edges. If G is bipartite, then show that $q \le \dfrac{p^2}{4}$.

Solution: Let G be a bipartite graph with p vertices and q edges with vertex partition $|V_1| = m$ and $|V_2| = n$

Then	$p = m + n$
and	$q \le mn.$
Therefore	$p^2 = (m+n)^2 = m^2 + 2mn + n^2$
and	$4q \le 4mn.$
Thus	$p^2 - 4q \ge m^2 + 2mn + n^2 - 4mn$
or	$p^2 - 4q \ge (m-n)^2$
Since	$(m-n)^2 \ge 0$, it follows that

$$p^2 - 4q \geq 0$$

or $$q \leq \frac{p^2}{4}.$$

Exercise

1. Draw all bipartite graphs with $p \leq 3$ vertices.

2. Give an example of a bipartite graph which is regular.

3. Give an example of a bipartite graph G with 5 vertices in which every vertex of G lies on a cycle.

4. Draw all complete bipartite graphs with $p \leq 3$ vertices.

5. Draw all complete bipartite graphs with 6 vertices.

6. Show that the following graphs are complete bipartite graphs.

i) 

ii) 

7. Prove that if G is a regular bipartite graph with partite sets V_1 and V_2, then $|V_1| = |V_2|$.

8. Show that every subgraph of a bipartite graph is bipartite.

9. Define
   i) a bipartite graph
   ii) a complete bipartite graph
   iii) a star

   with an example.

2.6 Characterization of Bipartite Graphs

We now present a chracterization of bipartite graphs in terms of its cycles.

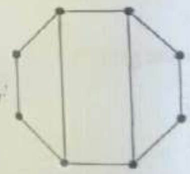
Theorem 2.8. A nontrivial graph is bipartite if and only if all of its cycles are even.

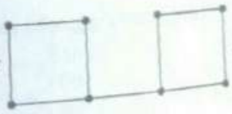
Proof: If G is a bipartite graph, then its vertex set V can be partitioned into two subsets V_1 and V_2 so that every edge of G joins a vertex of V_1 with a vertex of V_2. Suppose a cycle $C: v_1 v_2 \cdots v_n v_1$ is in G. Assume $v_1 \in V_1$. Then $v_2 \in V_2$, $v_3 \in V_1$, $v_4 \in V_2$, ..., $v_n \in V_2$. This implies that $n = 2s$ for some positive integer s. Thus C has even length.

Conversely suppose without loss of generality, that G is connected, for otherwise, we can consider the components of G separately. Take any vertex $v_1 \in V$. Let V_1 consist of v_1 and vertices at even distance from v_1 while $V_2 = V - V_1$. Since all cycles of G are even, every edge of G joins a vertex of V_1 with a vertex of V_2.

For further suppose there is an edge uv joining two vertices of V_1, then the union of two shortest paths from u and v_1 and from v_1 to v together with the edge uv contains an odd cycle. Thus a contradiction. Thus no two vertices of V_1 are adjacent. Similarly no two vertices of V_2 are adjacent. Thus G is a bipartite graph.

Example 29. Using characterization Theorem 2.8 show that the following graphs are bipartite graphs.

G_1: 

G_2: 

G_3: 

Solution:

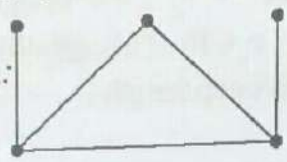
i) The graph G_1 has only even cycles. Therefore by Theor 2.8, G_1 is bipartite.

ii) The graph G_2 has only even cycles. Thus by Theorem 2.8, is bipartite.

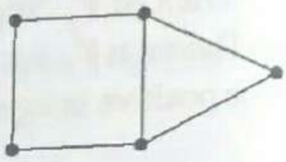
iii) The graph G_3 has only even cycles. Thus by Theorem 2.8 is bipartite.

Example 30. Using characterization Theorem 2.8, show that following graphs are not bipartite graphs.

G_1:



G_2:



Solution:

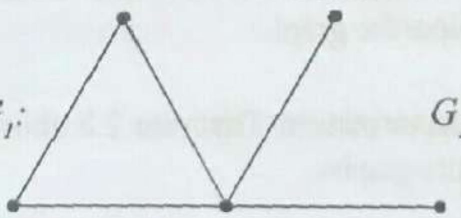
i) The graph G_1 contains a triangle. Thus by Theorem 2.8, G not bipartite.

ii) The graph G_2 contains a cycle of length 5 which is odd cyc Thus by Theorem 2.8, G_2 is not bipartite.

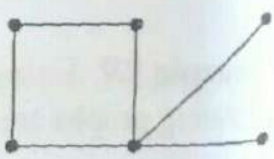
Exercise

1. Which of the following graphs are bipartite graphs ?

G_1:



G_2:



2. Which of the following graphs are bipartite graphs ?

i) K_2 ii) K_6

iii) $C_{2n}, n \geq 2$ iv) $C_{2n+1}, n \geq 1$

v) $P_{2n}, n \geq 2$ vi) $P_{2n+1}, n \geq 1$