On the Invariance of Maxwell’s Field Equations under Lorentz Transformations

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This paper shows that all the facts that seem to require Maxwell’s Field Equations to be invariant under the Lorentz transformations can be derived from assumptions different from what Einstein used. We start with Maxwell’s Field Equations and apply the relativity principle to them. With this approach, SRT is reformulated in a simple form that has its dynamical applications without using the LT and its kinematical contradictions.

Key words: Maxwell’s field equations, Lorentz Transformation, relativity principle

1. Introduction

Maxwell’s Field Equations (MFE’s) for the propagation of electromagnetic waves were not invariant under a Galilean transformation (GT), but were invariant under Lorentz transformations (LT). For invariance under LT, one must deal with three quantities: space, time and light speed. In Einstein’s approach for deriving the LT, [1] he connected the three quantities through a new velocity addition formula; i.e. for u along x

\[ v'_x = \frac{v_x - u}{1 - uv_x/c^2} \]  

(1)

Here Einstein used the basic definition for any velocity in frame S or S', i.e. :

\[ v_x = dx/dt, \quad v'_x = dx'/dt' \]  

(2)

and his second postulate, i.e. that for the special case of light,

\[ dx/dt = dx'/dt' = c \]  

(3)

This postulate was understood as a ‘measurement rule’.

Einstein proposed to modify all the laws of mechanics as well as electromagnetism to make them invariant under LT. This led directly to a demonstration of the essential role for LT and its kinematic effects in deriving the relativistic dynamical quantities and in the interpretation of relativistic phenomena. For instance, according to Eq. (1), one must accept the following explanation of invariant light speed: time itself has to slow down, and space must contract, to give always the same value as in Eq. (3). Thus Einstein introduced ‘relativity of simultaneity’ to physics.

This way for interpreting of the LT and its kinematic effects has long been questioned and misunderstood. Today, criticism [2] and paradox [3] still continue to receive attention. As a result, many physicists believe that a new interpretation, or even a theory alternative to SRT, may be needed [4]. Some physicists doubt that MFE’s are inherently invariant under LT (see e.g. [5]), and some others [6-8] doubt that the second postulate is necessary to build SRT. It seems unacceptable to keep the LT in as the starting point of SRT, especially if the LT is not necessary there and its role can be reduced to just a coordinate transformation [9a,b].

We have already shown that choosing a different set of postulates, one consisting of the relativity principle and the Lorentz force law (LFL), enables us to cancel the LT and its kinematical effects from the main body of SRT. In the present paper, the postulates chosen are the relativity principle and the MFE’s. The MFE’s stand in place of Einstein’s ‘relativity of simultaneity’. In contrast to Einstein’s LT with its kinematical effects [1], the LT produced by our alternative method is simply a neutral transformation, containing no physical significance.

2. Electro-Magnetic Fields, Current-Charge Density, and Lorentz-Transformations

The MFE’s in frame S may be expressed as

\[ \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \]  

(4a)

\[ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \partial E / \partial t + \mathbf{\mu_0 J} \]  

(4b)

\[ \nabla \cdot \mathbf{B} = 0 \]  

(4c)

\[ \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \]  

(4d)

To understand the parameter c in Eq. (4b) as a constant light speed, we must define the light speed c as a function of the universal constant \( \mu_0 \) in Eq. (4b). In the absence of charge \( q \) (i.e. in free space), the MFE’s lead to the well-known wave equation. This wave equation allows us to calculate the speed of electromagnetic radiation, which includes visible light. The speed of light can be written in terms of pure electromagnetic constants as

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]  

Now by applying the relativity principle to Eqs. (4), they will preserve their form in frame S' moving with velocity \( \mathbf{u} \) parallel to their common x axis, and are expressed as follows:

\[ \nabla' \cdot \mathbf{E}' = \rho' / \varepsilon_0 \]  

(5a)

\[ \nabla' \times \mathbf{B}' = \partial \mathbf{E}' / \partial t' + \mathbf{\mu_0 J}' \]  

(5b)

\[ \nabla' \cdot \mathbf{B}' = 0 \]  

(5c)

\[ \nabla' \times \mathbf{E}' = -\partial \mathbf{B}' / \partial t' \]  

(5d)

where \( \rho', \mathbf{J}', \mathbf{J}' \) are the relativistic charge and current density vectors in frames S and S', respectively. It is obvious that in frame S' the field vectors \( \mathbf{E}' \) and \( \mathbf{B}' \) are propagated in free-space with the speed \( c' = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \). But we know that \( \varepsilon_0 \) and \( \mu_0 \) have the...
same value in all frames, i.e. \( \varepsilon_0 \) and \( \mu_0 \) are universal constants, so \( c' \) must equal \( c \). In conclusion, the laws of electrodynamics (since a number of expressions of physical laws contain \( \varepsilon_0 \) and \( \mu_0 \)), together with the relativity principle, imply the invariance of light speed. So the question by this formalism is not in the well-known concept of invariant light speed, but the question is: What are the properties that light has to travel at \( v = c \) in passing from one frame to another? Taking the \( x \)-component of Eq. (4b) and writing Eq. (4a) in terms of Cartesian components, we have

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \partial E_x / c^2 \partial t + \mu_0 \partial J_x \quad (6a)
\]

\[
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho / \varepsilon_0 \quad (6b)
\]

Multiplying Eq. (6a) by an arbitrary scalar factor \( \gamma \) and (6b) by \( \gamma v / c^2 \), and then subtracting, we get:

\[
\frac{\partial}{\partial y} \gamma (B_z - uE_y / c^2) - \frac{\partial}{\partial z} \gamma (B_y + uE_z / c^2) = \gamma \left( \frac{\partial}{\partial t} + \frac{u}{c^2} \frac{\partial}{\partial x} \right) \left( J_x - u \rho \right)
\]

Comparing the last relation with the \( x \)-component of Eq. (5b), we have

\[
E_x = B_x \quad B_y = \gamma (B_y + uE_z / c^2) \quad B_z = \gamma (B_z - uE_y / c^2) \quad (7a)
\]

\[
\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \frac{u}{c^2} \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \quad (8a)
\]

and

\[
J_x' = \gamma (J_x - u \rho) \quad (9a)
\]

Now multiplying Eq. (6a) by \( \gamma v / c^2 \) and Eq. (6b) by \( \gamma \), and then subtracting, we have

\[
\gamma \left( \frac{\partial}{\partial x} + \frac{u}{c^2} \frac{\partial}{\partial t} \right) E_x + \frac{\partial}{\partial y} \gamma (E_y - uB_z) + \frac{\partial}{\partial z} \gamma (E_z + uB_y) = \frac{\rho}{\varepsilon_0} \left( 1 - u^2 / c^2 \right)
\]

Comparing the last relation with Eq. (5a), we have

\[
E_x' = \gamma (E_x - uB_z) \quad (7b) \quad E_y' = \gamma (E_y + uB_z) \quad (7b)
\]

\[
\frac{\partial}{\partial x'} = \gamma \left( \frac{\partial}{\partial x} + u \frac{\partial}{\partial t} / c^2 \right) \quad (8b)
\]

and

\[
\rho' = \rho - uJ_x / c^2 \quad (9b)
\]

The Eqs. (8) are the differential LT. The scalar factor \( \gamma \) can be fixed by applying the relativity principle on Eq. (7a). So Eq. (7a) could be written in frame \( S' \) as

\[
B_z = \gamma (B_x' + uE_y' / c^2) \quad (10)
\]

Hence we can substitute from Eqs. (7a) and (7b) into Eq. (10) and obtain

\[
B_z = \gamma \left[ \gamma (B_x - uE_y / c^2) + u \gamma (E_y - uB_z) / c^2 \right] = \gamma^2 (1 - u^2 / c^2) B_z
\]

This equation is a algebraic identity; i.e.,

\[
\gamma^2 (1 - u^2 / c^2) = 1 \quad (11a) \quad \text{or} \quad \gamma = 1 / \sqrt{1 - u^2 / c^2} \quad (11b)
\]

Starting now from the \( y \)-component of Eq. (4b) and using Eq. (11a), we get

\[
\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} = \gamma \left[ \frac{\partial}{\partial t} + \frac{u}{c^2} \frac{\partial}{\partial x} \right] (E_y - uB_z) + \mu_0 J_y
\]

Adding and subtracting the two terms

\[
\gamma v^2 \frac{\partial E_y}{\partial c^2} \partial x, \quad \gamma v^2 \frac{\partial B_z}{\partial c^2} \partial t
\]

We obtain

\[
\frac{\partial B_y}{\partial z} = \gamma \left[ \frac{\partial}{\partial t} + \frac{u}{c^2} \frac{\partial}{\partial x} \right] (E_z - uB_y) + \mu_0 J_y
\]

Comparing the last relation with the \( y \)-component of Eq. (5b), we deduce

\[
B'_y = B_y \quad (7c)
\]

and

\[
J'_y = J_y \quad (9c)
\]

In a similar way, starting from the \( z \)-component of Eq. (4b), we obtain

\[
J'_z = J_z \quad (9d)
\]

The electromagnetic charge density \( \rho \) for a particle of charge \( q \) is \( \rho = q \delta^3 (x - x') \). It implies the conventional definition of current density \( J = qv \delta^3 (x - x') = q \rho v \). Starting from the definition of \( J \) in frames \( S \) and \( S' \) as \( J = \rho v \mathbf{v}' \), \( J' = \rho' \mathbf{v}' \), and using Eqs. (9), one gets the relativistic velocity transformations, i.e.

\[
\mathbf{v}' = \frac{v_x - u}{1 - uv_x / c^2}, \quad \mathbf{v}' = \frac{v_y / \gamma}{(1 - uv_x / c^2)} = \frac{v_z / \gamma}{(1 - uv_x / c^2)} \quad (12)
\]

As it is shown in paper [9b], depending on Eqs. (12) and Eqs. (9), one can derive the relativistic charge density in frame \( S \) and \( S' \) respectively:

\[
\rho = \rho_0 / \sqrt{1 - u^2 / c^2}, \quad \rho' = \rho_0 / \sqrt{1 - u^2 / c^2} \quad (13)
\]

where \( \rho_0 \) is the electrostatic charge density measured when the charge is at rest relative to the observer.

Many physicists have demonstrated the invariance of MFE’s under LT, but they have not used our approach. In our approach, we begin with the Maxwell equations, Eqs. (4), and by applying the relativity principle we get the Lorentz contracted charge density containing \( \rho \) defined as Eqs. (13), without using the hypothesis proposed by Lorantz-Fitzgerlad as well as with-
out using the length contraction. Thus none of Einstein’s results change; it is only the approach that changes.

3. On Light-Speed Invariance

Although the objections by L.B. Okun [10] state that the proper definition of relativistic momentum and energy are \( P = m_0 \gamma p \) and \( \varepsilon = m_0 \gamma c^2 \), the traditional definition of momentum, i.e. \( P = mv \), combined with Eqs. (12) gives all the relativistic expressions and relativistic transformation relations [9a]. We obtain also in this work the same results as in paper [9a] concerning momentum, energy and mass; especially, we get the invariance of the 4-vector momentum-energy relation, i.e.

\[
E^2 - c^2 p^2 = E'^2 - c^2 p'^2
\]  

(14)

As is well known, SRT is based on two postulates [1]. The first postulate, the relativity principle, is evident. But the second one, light-speed invariance, is difficult to accept. The great question has been: “Why does light speed appear to be the same for every one?” SRT did not actually answer this question, but instead dealt with it by simply asserting Eq. (3). To deal with this question in the present formalism, it is essential to insist that the invariance of light speed should be derivable from the basic physical relations such as Eq. (14), instead of the second postulate. Eq. (14) also expresses the same fact that light travels at \( c \) in all systems. So the properties that light has to travel at \( c \) in passing from one frame to another depend on the properties of light itself, and not on the properties of some medium such as space–time, as shown below:

As it is known, the total energy and the total momentum of the wave light satisfy [9a]:

\[
E = cp \quad (15a) \quad E' = cP' \quad (15b)
\]

And from Eqs. (15) we see

\[
c = E / P = E' / P'
\]

(16)

As we see, the invariance of light speed, Eq. (16), is the result of the response of energy-momentum to the relative motion through the change in energy-momentum to satisfy always the invariance Eqs. (16).

A similar result can be obtained if the light is considered through its properties frequency \( \omega \) and wave number \( k \), which relates to relations

\[
E = \hbar \omega, \quad P = \hbar k
\]

(17)

Using Eq. (17) in Eqs. (14) and (15a), gives

\[
\omega^2 - c^2 k^2 = \omega'^2 - c^2 k'^2 \quad (18a) \quad \text{and} \quad \omega = c k
\]

(18b)

Substituting Eq. (18b) in Eq. (18a), yields

\[
\omega^2 - c^2 k^2 = 0 \quad (19a) \quad \text{i.e.} \quad \omega'^2 - c^2 k'^2 = 0 \quad (19b)
\]

and

\[
c = \omega / k = \omega' / k'
\]

(20)

As we see, the invariance of light speed, Eq. (20), is the result of the response of frequency-wave number to the relative motion through the change in frequency-wave number to satisfy always the invariance Eqs. (20). So the light speed could be reinterpreted by using wave light’s dynamical quantities (energy, momentum, frequency, and wave number) and not the kinematics of LT as in Eq. (3).

4. Conclusion

By comparing the results in papers [9a],[9b] with the results in this paper, we see that there is no physical distinction between the LFL and MFE’s. So MFE’s should govern the relativistic electromagnetic phenomena exactly as LFL do. Therefore our aim in this work was to extend the relativity principle to hold true for MFE’s as held on LFL. Hence we have got the same results as the results in papers [9a],[9b].

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References
