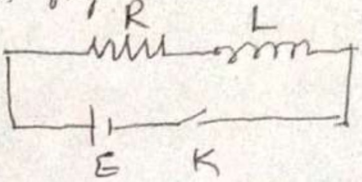
B.sc II^{nd} Sem. (BCS)

Transient currents

Growth of current in a Circuit containing RL.

consider a circuit having an Inductance L and a resistance R. connected in series to a cell. of steady emf E as shown in fig



when the key K is pressed then is a grasual growth of current in the circuit from Zero to maximum value I_0.

Let I be the instanteneous current at any instant

Then the induced back emf $e = -L\frac{dI}{dt}$

$$E = RI + L\frac{dI}{dt} \quad \text{--- (1)}$$

when the current reaches the maximum value I_0 the back emf, $L\frac{dI}{dt} = 0$.

Hence $\quad E = RI_0 \quad$ ——— (2)

Substuting the value of E in Eqn. (1)

$$RI_0 = RI + L\frac{dI}{dt} \quad \text{or} \quad R(I_0 - I) = L\frac{dI}{dt}$$

or $\frac{dI}{I_0 - I} = \frac{R}{L}dt$

Integrating $-\log(I_0 - I) = \frac{R}{L}t + c \quad$ — (3)

where c is the constant of integration.

when $t=0$, $I=0$, $\therefore -\log_e I_0 = c$

Substuting the value of c in Eqn (3)

$$-\log_e(I_0 - I) = \frac{R}{L}t - \log_e I_0.$$

or $\log_e(I_0 - I) - \log_e I_0 = -\frac{R}{L}t$

$$\log_e\frac{(I_0 - I)}{I_0} = -\frac{R}{L}t$$

$$\frac{I_0 - I}{I_0} = e^{-\frac{R}{L}t} \quad \text{or} \quad 1 - \frac{I}{I_0} = e^{-\frac{R}{L}t}.$$

$$I = I_0(1 - e^{-R/L\,t}) \quad \text{—— (4)}$$

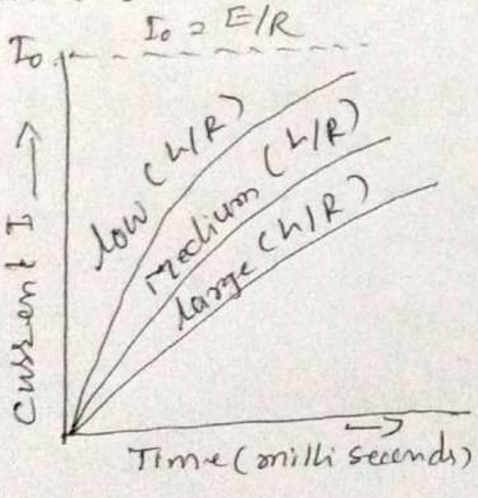
Equn. (4) gives the value of the instanteneous current in the LR Circuit, The quantity (L/R) is called the Time constant of the Circuit

If $\frac{L}{R} = t$, $I = I_0(1 - e^{-1}) = I_0\left[1 - \frac{1}{e}\right] = 0.632\,I_0$

Thus the Time constant L/R of a L-R Circuit is the time taken by the current to grow from zero to 0.632 times the steady maximum value of current in the circuit

similarly when $t = \frac{2L}{R}, \frac{3L}{R}$ ... the value of current will be 0.8647, 0.9502 .... of the final maximum current when $t = 0$, $I = 0$ and when $t \to \infty$, $I = I_0$.

Greater the value of L/R, longer is the time taken by the current I to reach its maximum value below fig.



Decay of current in a Circuit containing L and R

When the Circuit is broken, an induced emf equal to $-L\frac{dI}{dt}$ is again produced in the inductance L, and it slow down the rate of decay of the current, The current in the circuit decays from maximum value I_0 to zero,. During the decay, let I be the current at time t In this case $E = 0$, The emf, equation for decay of current is

$$0 = RI + L\frac{dI}{dt} \quad \text{—— (1)}$$

$$\frac{dI}{I} = -R/L\,dt$$

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Integrating $\log_e I = -\frac{R}{L} t + c$ where c is constant

when $t=0$, $I = I_0$ $\therefore \log_e I_0 = c$

$\therefore \log_e I = -\frac{R}{L} t + \log_e I_0$

or $\log_e \frac{I}{I_0} = -\frac{R}{L} t$

$$\frac{I}{I_0} = e^{-\frac{R}{L} t}$$

$$I = I_0 e^{-\frac{R}{L} t} \quad \text{——— (2)}$$

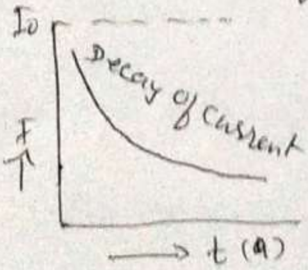
Eqn. (2) represents the current at any instant t during decay. A graph between current and time is shown in fig.

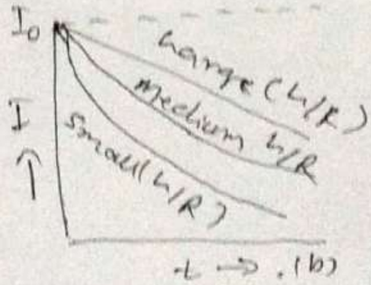
when $t = L/R$, $I = I_0 e^{-1} = \frac{1}{e} I_0 = 0.365 I_0$.

$t = \frac{2L}{R}$, $I = I_0 e^{-2} = 0.1035 I_0$.

$t = \frac{3L}{R}$, $I = I_0 e^{-3} = 0.05 I_0$.

Therefore the time constant $\frac{L}{R}$ of a R-L circuit may also be defined as the time in which the current in the circuit falls to $1/e$ of its maximum value when external source of emf is removed.

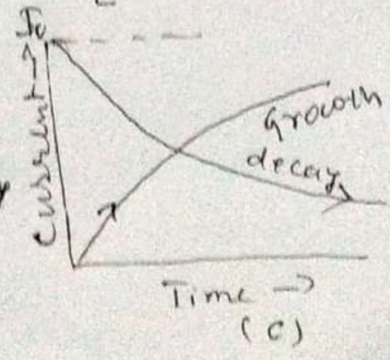




The rate of decay of current is

$$dI/dt = \frac{R}{L} I_0 e^{-\left(\frac{R}{L}\right)t} = \frac{R}{L} I$$

Thus it is clear that greater the ratio R/L or smaller the time constant L/R, the more rapidly does the current die away. fig (c) shows growth and decay curves are complementary



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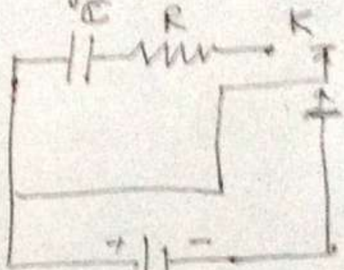
Charging and discharging of a capacitor through a Resistor

a) Growth of charge: A capacitor C and a Resistor R are connected to a cell of emf E through a Morse Key (shown in fig), when the key is pressed a momentary current I flows through R. At any instant t, let Q be the charge on the capacitor of capacitance C



P.D. across capacitor $= Q/C$

P.D. across resistor $= RI$

The emf equation of the circuit is

$$E = (Q/C) + RI$$

$$E = (Q/C) + R\left(\frac{dQ}{dt}\right) \quad \left[\because I = \frac{dQ}{dt}\right]$$

The capacitor continues getting charged till it attains maximum charge Q_0. At that instant $I = \frac{dQ}{dt} = 0$. The P.D. ~~across~~ across the capacitor is $E = Q_0/C$

i.e. when $Q = Q_0$, $\frac{dQ}{dt} = 0$, $E = \frac{Q_0}{C}$

$$\frac{Q_0}{C} = \frac{Q}{C} + R\frac{dQ}{dt}$$

$$(Q_0 - Q) = CR\frac{dQ}{dt}$$

$$\frac{dQ}{(Q_0 - Q)} = \frac{dt}{CR} \qquad \text{——— (2)}$$

Integrating $-\log_e(Q_0 - Q) = \frac{t}{CR} + K$.

where K is a constant.

when $t = 0$, $Q = 0$ $\therefore -\log_e Q_0 = K$.

$$\therefore -\log_e(Q_0 - Q) = \frac{t}{CR} - \log_e Q_0.$$

$$\log_e(Q_0 - Q) = -\frac{t}{CR} + \log_e Q_0$$

$$\log_e(Q_0 - Q) - \log_e Q_0 = -\frac{t}{CR}$$

$$\log_e\left[\frac{Q_0 - Q}{Q_0}\right] = -\frac{t}{CR}$$

$$\frac{Q_0 - Q}{Q_0} = e^{-t/CR} \quad \text{or} \quad 1 - \frac{Q}{Q_0} = e^{-t/CR}$$

$$Q = Q_0(1 - e^{-t/CR}) \qquad \text{—— (3)}$$

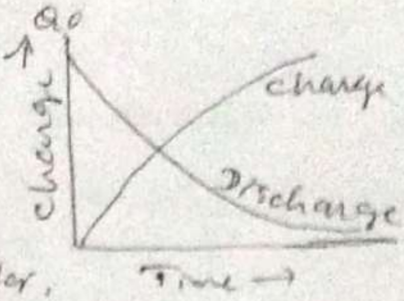
The term CR is called time constant of the circuit. At the end of time $t = CR$, $Q = Q_0(1 - e^{-1}) = 0.632 Q_0$

Thus the time constant may be defined as the time taken by the capacitor to get charged to 0.632 times its maximum value.

The growth of charge is shown in fig.

The rate of growth of charge is

$$\frac{dQ}{dt} = \frac{Q_0}{CR} e^{-\frac{t}{CR}} = \frac{1}{CR}(Q_0 - Q)$$



Thus it is seen that smaller the product CR, the more rapidly does the charge grow on the capacitor.

The rate of growth of the charge is rapid in the begining and it becomes less and less as the charge approches nearer and nearer the steady value.

b) <u>Decay of charge (Discharging of a capacitor through Resistance)</u>

Let the capacitor having charge Q_0 be now discharged by releasing the morse Key (above fig). The charge flows out of the capaciter and this constitutes a current. In this case $E = 0$.

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0 \qquad \text{—— (1)}$$

$$\frac{dQ}{Q} = -\frac{1}{CR}\,dt$$

Integrating $\log_e Q = -\frac{t}{CR} + K$, where K is constant.

when $t=0$, $Q = Q_0$ $\therefore \log_e Q_0 = K$.

$$\log_e Q = -\frac{t}{CR} + \log_e Q_0$$

or $\log_e \frac{Q}{Q_0} = -\frac{t}{CR}$ or $\frac{Q}{Q_0} = e^{-t/CR}$

$$Q = Q_0 e^{-t/CR} \quad \text{——— (2)}$$

This shows that the charge in the capacitor decays Exponentially and becomes zero after infinite interval of time. (fig shown above)

The rate of discharge is

$$I = \frac{dQ}{dt} = -\frac{Q_0}{CR} e^{-t/CR} = -\frac{Q}{CR} \quad \text{——— (3)}$$

Thus smaller the time constant CR, the quicker is the discharge of the capacitor.

In eqn (2) if we put $t = CR$ then $Q = Q_0 e^{-1} = 0.368 Q_0$.

Hence time constant may also be defined as the time taken by the current to fall from maximum to 0.368 of its maximum value.

Growth of charge in a circuit with Inductance, Capacitance and Resistance :-

Consider a circuit containing an inductance L, capacitance C and resistance R joined in series to a cell of emf. E (in fig) when the key K is pressed, the capacitor is charged, Let Q be the charge on the capacitor and I the current in the circuit at an instant t during charging. Then p.d. across the capacitor is Q/C and the self induced emf in the inductance coil is $L(dI/dt)$, both being opposite in direction of E, The pd across the Resistance R is RI

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The equation of emf is

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E \quad \text{——} (1)$$

But. $I = \frac{dQ}{dt}$ and $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$

$$\therefore \; L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

or $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q - CE}{LC} = 0.$

putting $\frac{R}{L} = 2b$. and $\frac{1}{LC} = k^2$ we have.

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + k^2(Q - CE) = 0 \quad \text{—} (2)$$

Let $x = Q - CE$, Then $\frac{dx}{dt} = \frac{dQ}{dt}$ and $\frac{d^2x}{dt^2} = \frac{d^2Q}{dt^2}$

Eqn (2) becomes $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0$ — (3)

Hence the most general solution of Eqn (3) is

$$x = Ae^{[-b + \sqrt{(b^2-k^2)}]t} + Be^{[-b - \sqrt{(b^2-k^2)}]t}$$

Now $CE = Q_0$ = final steady charge on the capacitor

$\therefore x = Q - CE = Q - Q_0$

Hence $Q - Q_0 = Ae^{[-b + \sqrt{(b^2-k^2)}]t} + Be^{[-b - \sqrt{(b^2-k^2)}]t}$

or $Q = Q_0 + Ae^{[-b + \sqrt{(b^2-k^2)}]t} + Be^{[-b - \sqrt{(b^2-k^2)}]t}$ — (4)

Using initial conditions

at $t = 0$, $Q = 0$.

$\therefore 0 = Q_0 + (A+B)$ or $A + B = -Q_0$ —— (5)

$$\frac{dQ}{dt} = A\left(-b+\sqrt{(b^2-K^2)}\right] e^{[-b+\sqrt{(b^2-K^2)}]t} + B\left[-b-\sqrt{(b^2-K^2)}\right) e^{[-b-\sqrt{(b^2-K^2)}]t}$$

At $t=0$, $\frac{dQ}{dt} = 0$

$$0 = A\left[-b+\sqrt{(b^2-K^2)}\right] + B\left[-b-\sqrt{(b^2-K^2)}\right)$$

$$\sqrt{(b^2-K^2)}\,[A-B] = b(A+B) = -bQ_0$$

or $$A-B = -\frac{Q_0 b}{\sqrt{(b^2-K^2)}} \quad \text{—— (6)}$$

Solving Eqn (5) and (6)

$$A = -\tfrac{1}{2}Q_0\left[1+\frac{b}{\sqrt{(b^2-K^2)}}\right] \quad \text{— (7)}$$

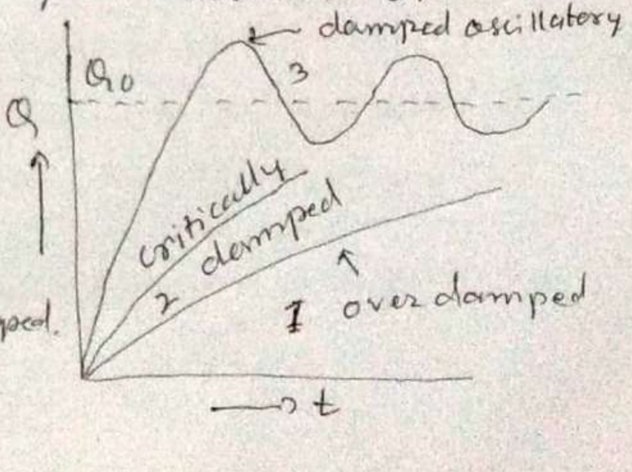
$$B = -\tfrac{1}{2}Q_0\left[1-\frac{b}{\sqrt{(b^2-K^2)}}\right] \quad \text{— (8)}$$

Substuting the values of A and B in Eqn (4)

$$Q = Q_0 - \tfrac{1}{2}Q_0\, e^{-bt}\left[\left(1+\frac{b}{\sqrt{(b^2-K^2)}}\right] e^{\sqrt{(b^2-K^2)}t} + \left[1-\frac{b}{\sqrt{(b^2-K^2)}}\right] e^{-\sqrt{(b^2-K^2)}.t}\right] \quad \text{-- (9)}$$

Case I :- If $b^2 > K^2$, $\sqrt{(b^2-K^2)}$ is real. The charge on the capacitor grows exponentially. with time and. attains the maximum value Q_0 asymptotically. (curve 1. of fig) The charge is known as over damped or dead beat.

Case II : If $b^2 = K^2$, the charge rises to maximum value Q_0 in a short time (curve 2 in fig) Such a charge is called critically damped.



Case III :- If $b^2 < K^2$, $\sqrt{(b^2-K^2)}$ is imaginary.

Let $\sqrt{b^2-K^2} = i\omega$ where $i = \sqrt{-1}$ and $\omega = \sqrt{K^2-b^2}$

Eqn. (9) may be written as.

$$Q = Q_0 - \tfrac{1}{2}Q_0 e^{-bt}\left[\left(1+\frac{b}{i\omega}\right)e^{i\omega t} + \left(1-\frac{b}{i\omega}\right)e^{-i\omega t}\right]$$

$$Q = Q_0 - Q_0 e^{-bt}\left[\frac{e^{i\omega t}+e^{-i\omega t}}{2} + \frac{b}{\omega}\frac{(e^{i\omega t}-e^{-i\omega t})}{2i}\right]$$

$$Q = Q_0 - Q_0 e^{-bt}\left[\cos\omega t + \frac{b}{\omega}\sin\omega t\right]$$

$$Q = Q_0\left[1 - \frac{e^{-bt}}{\omega}(\omega\cos\omega t + b\sin\omega t)\right]$$

Let $\omega = K\sin\alpha$ and $b = K\cos\alpha$ so that $\tan\alpha = \omega/b$.

$$Q = Q_0\left[1 - \frac{e^{-bt}}{\omega}(K\sin\alpha\cos\omega t + K\cos\alpha\sin\omega t)\right]$$

or $$Q = Q_0\left[1 - \frac{Ke^{-bt}}{\omega}\sin(\omega t+\alpha)\right] \quad \text{——— (10)}$$

$$Q = Q_0\left[1 - \frac{e^{-\frac{R}{2L}t}\sqrt{\frac{1}{LC}}}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin\left[\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t+\alpha\right]\right]$$

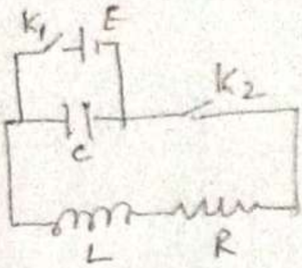
This equation represents a damped oscillatory charge as shown by the curve(3). The charge oscillates above and below Q_0 till it finally settles down to Q_0 value. The frequency of oscillation in the circuit is given by

$$\nu = \frac{\omega}{2\pi} = \frac{\sqrt{K^2-b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

when $R=0$, $\nu = \frac{1}{2\pi\sqrt{LC}}$

Discharge of a capacitor through an Inductor and Resistor in Series (Decay of charge in LCR Circuit)

Consider a circuit containing a capacitor of capacitance C, an inductance L and a Resistor R joined in series [fig 2]. E is a cell, K_2 is kept open. The capacitor is charged to maximum charge Q_0 by closing the key K_1. on opening K_1 and closing K_2 the capacitor discharges R. Let I be the current in the circuit and Q be the charge in the ~~circuit~~ capacitor at any instant during discharge. The circuit equation then is



$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0.$$

But. $I = \frac{dQ}{dt}$ and $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$

$$\therefore L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0 \quad \text{———} (1)$$

Let $\frac{R}{L} = 2b$ and $\frac{1}{LC} = K^2$, then

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + K^2Q = 0. \quad \text{———} (2)$$

The general solution of this eqn is

$$Q = Ae^{(-b+\sqrt{b^2-K^2})t} + Be^{(-b-\sqrt{b^2-K^2})t} \quad \text{——} (3)$$

where A and B are arbitrary constants.

when $t=0$, $Q = Q_0$ and from eqn (3) $A+B = Q_0$ —— (4)

$$\frac{dQ}{dt} = A(-b+\sqrt{b^2-K^2})e^{(-b+\sqrt{b^2-K^2})t}$$
$$+ B(-b-\sqrt{b^2-K^2})e^{(-b-\sqrt{b^2-K^2})t}$$

when $t=0$, $\frac{dQ}{dt} = 0$

$$\therefore A(-b+\sqrt{b^2-K^2}) + B(-b-\sqrt{b^2-K^2}) = 0.$$
$$-b(A+B) + \sqrt{b^2-K^2}(A-B) = 0.$$
$$-bQ_0 + \sqrt{b^2-K^2}(A-B) = 0.$$

$$\therefore \quad A - B = \frac{bQ_0}{\sqrt{b^2-K^2}} \quad \text{——} \ (5)$$

From Eqn (4) and (5) we get

$$A = \tfrac{1}{2} Q_0 \left[1 + \frac{b}{\sqrt{(b^2-K^2)}}\right] \text{ and } B = \tfrac{1}{2} Q_0 \left[1 - \frac{b}{\sqrt{b^2-K^2}}\right]$$

putting these values of A and B in Eqn (3) we get

$$Q = \tfrac{1}{2} Q_0 e^{-bt} \left[\left[1 + \frac{b}{\sqrt{b^2-K^2)}}\right] e^{\sqrt{(b^2-K^2)}t} + \left[1 - \frac{b}{\sqrt{(b^2-K^2)}}\right] e^{-\sqrt{(b^2-K^2)}t} \right] \quad \text{——} \ (6)$$

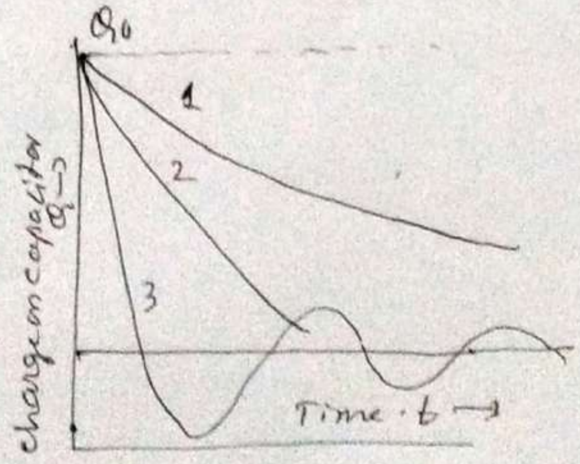
Case I :- If $b^2 > K^2$, $\sqrt{b^2-K^2}$ is real and positive and the charge of the capacitor decays exponentially, becomes zero asymptotically (curve 1.) This discharge is known as over damped non-oscillatory or dead beat.

Case II :- when $b^2 = K^2$, $Q = Q_0 (1 + bt) e^{-bt}$

This represents a non-oscillatory, discharge, This discharge is known as critically damped (curve 2) The charge decreases to zero exponentially in a short time

Case III :- If $b^2 < K^2$, $\sqrt{(b^2-K^2)}$ is imaginary

$\sqrt{b^2-K^2} = i\omega$ where $\omega = \sqrt{K^2-b^2}$



$$\therefore Q = \tfrac{1}{2} Q_0 e^{-bt}\left[\left(1+\frac{b}{i\omega}\right)e^{i\omega t} + \left(1-\frac{b}{i\omega}\right)e^{-i\omega t}\right]$$

$$= Q_0 e^{-bt}\left[\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right) + \frac{b}{\omega}\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)\right]$$

$$= \frac{Q_0 e^{-bt}}{\omega}(\omega \cos\omega t + b \sin\omega t)$$

Let $\omega = K \sin\alpha$ and $b = K\cos\alpha$ so that $\tan\alpha = \frac{\omega}{b}$.

$$Q = \frac{Q_0 e^{-bt} K}{\omega}(\cos\omega t \sin\alpha + \cos\alpha \sin\omega t)$$

$$= \frac{Q_0 e^{-bt} K}{\omega}\sin(\omega t + \alpha)$$

$$Q = \frac{Q_0 e^{-\frac{R}{2L}t}}{\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\sqrt{LC}} \sin\left[\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\,t + \alpha\right] \quad -(7)$$

This equation represents a damped oscillatory charge as shown in fig (3). The charge oscillates above and below zero till it finally settles down to zero value.

The frequency of oscillation in the circuit is given by

$$\nu = \frac{\omega}{2\pi} = \frac{\sqrt{K^2-b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

when $R = 0$, $\nu = \frac{1}{2\pi\sqrt{LC}}$

The condition for oscillatory discharge is

$$\frac{R^2}{4L^2} < \frac{1}{LC} \text{ or } R < 2\sqrt{\frac{L}{C}}$$